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TOPOLOGICAL PROPERTIES OF MINIMAL SURFACES

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SUMMARY

This thesis describes examples which answer two questions posed by Meeks about the topology of minimal surfaces.

Question 1 [Meeks1, conjecture 5][Meeks2, Problem 1]. Given a set Γ of disjoint smooth Jordan curves on the standard 2-sphere S^2, such that Γ bounds two homeomorphic embedded compact connected minimal surfaces F and G in B^3, is there an isotopy of B^3 fixing Γ and taking F to G?

Meeks has shown that such surfaces always split B^3 into two handlebodies; it then follows that such an isotopy exists if Γ consists of a single curve or if F and G are annuli [Meeks2, Theorem 2]. We give two examples where F and G are not isotopic: in one example F and G are planar domains with three boundary components and in the other they have genus one and two boundary components.

Question 2 [Meeks1, conjecture 2][Nitschel, §910(b)]. Can a Jordan curve on the boundary of a convex set in $\mathbb{R}^3$ bound a minimal disc that is not embedded?

Meeks and Yau have proved that such a disc is embedded under the assumption that it solves the problem of least area for its boundary [MY1, Theorem 2]. We give an example that shows this assumption is necessary.

Our examples can be described informally using the "bridge principle," a heuristic method for constructing minimal surfaces which was introduced by Courant [Courant, Lemma 3.3] and Lévy [Lévy, Chapter I, Section 6]. A method for making such examples rigorous was given by Meeks and Yau [MY2, Theorem 7], and we include an exposition of the results of theirs that we need.

to Rae Sustins

my mathematics teacher at school

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The Mathematics Institute at Warwick has been a most stimulating environment in which to work and I am grateful to all those of its staff, students and friends with whom I have discussed mathematics.

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Most of all I wish to thank D.B.A. Epstein for his patient and inspiring supervision.

DECLARATION

The research in this thesis is due to the author alone except where results are attributed to other people.

A paper the contents of which are essentially those of Chapter 5 has been submitted for publication under the title "Two topological examples in minimal surface theory."

INTRODUCTION

"Ce problème, qui doit compter au nombre des plus intéressants que l'Experience ait jamais posés aux Géomètres, est aussi un de ceux dont les progrès sont le plus étroitement liés à ceux de l'Analyse moderne."
Darboux on the Plateau problem [Darboux, Part I, p.601].

In his paper on "The topological uniqueness of minimal surfaces in three dimensional Euclidean space" [Meeks2], Meeks states the following conjecture as Problem 1.

"Suppose Γ is a collection of disjoint Jordan curves on $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. If M_1 and M_2 are embedded homeomorphic compact connected minimal surfaces in the unit ball B which have boundary Γ, then M_1 and M_2 are isotopic in B."

This is the same as an affirmative answer to Question 1 of our Chapter 5. Meeks states that this conjecture implies another conjecture, the "topological uniqueness" of a properly embedded complete minimal surface of finite genus in $\mathbb{R}^3$; his line of reasoning is to intersect such a minimal surface with a large sphere, cutting it into a central piece and some annular ends, and attempt to show that both the central piece and the ends are standardly embedded. He proves the Problem 1 conjecture if either Γ is a single Jordan curve or M_1 and M_2 are annuli. The same conjecture in a slightly stronger form is Conjecture 5 of an earlier problem list [Meeks1] in which he states that it implies the "topological uniqueness" of a complete embedded minimal surface in a flat 3-torus. In Chapter 5 we give examples to show that the conjecture is false in general.

For a rectifiable Jordan curve Γ on the standard S^2 in $\mathbb{R}^3$ Meeks and Yau have proved that a disc of least area among discs spanning Γ is embedded. It is natural to ask whether this is the case for any disc spanning Γ which is a minimal surface [Nitschel, §910(b)] [Meeks1, conjecture 2]. In Chapter 5 we give an example of a Jordan curve Γ on S^2 which bounds an immersed stable minimal disc that is not embedded.

Our examples are fairly simple to describe heuristically using the "bridge principle" for minimal surfaces. This is a property that one expects to hold of minimal surfaces by analogy with soap films. In rough terms it is as follows. Suppose that F and G are stable minimal surfaces in $\mathbb{R}^3$ and ∂F does not intersect ∂G (Figure 0(a)). Let γ be an embedded arc running from a point of ∂F to a point of ∂G (Figure 0(b)). Modify $\partial F \cup \partial G$ to form a new set Δ of Jordan curves by erasing short arcs of ∂F and ∂G near the ends of γ and adding two arcs close to γ (Figure 0(c)). Then Δ bounds a new minimal surface consisting of surfaces close to F and G together with a thin "bridge" along γ.

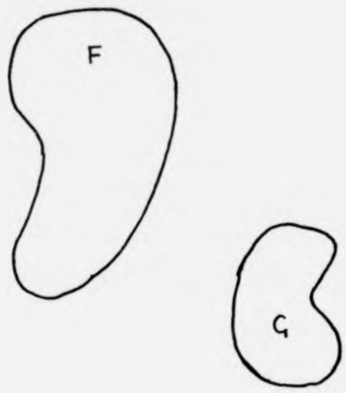

Figure O(a).

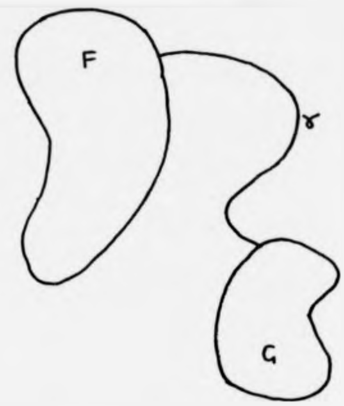

Figure O(b).

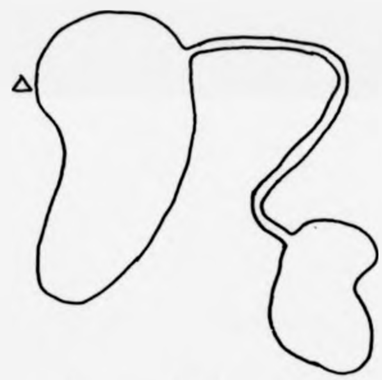

Figure O(c).

The bridge principle was introduced independently by Courant [Courant, Lemma 3.3] and Lévy [Lévy, Chapter I, Section 6]. Lévy outlines a proof which somewhat resembles that of Meeks and Yau as given in this thesis; several of his steps are hard to make rigorous. Courant asserts that a proof may be gives by the methods of Chapter VI of his book; an attempt was made by Kruskal to write down a proof on these lines [Kruskal] but Nitsche pointed out that it contained a serious mistake [Nitsche2, p.396]. (The mistake is [Kruskal, p.310, lines 9-5 from below]. Kruskal regarded $\partial F \cup \partial G \cup \gamma$

as a limiting case of a set of Jordan curves such as Δ. He discussed a minimal surface "spanning" $\partial F \cup \partial G \cup \gamma$ and attempted to show that it was the union of surfaces spanning ∂F and ∂G.) A proof of a form of the bridge principle has recently been given by Meeks and Yau [MY2, Theorem 7], and it is their method that is followed in this thesis. They refer to another recent proof by Almgren and Solomon, of which no details have yet become available.

Meeks and Yau discuss the bridge principle with the techniques of their work on the existence of embedded minimal surfaces in three-manifolds. The idea is to construct a submanifold M of $\mathbb{R}^3$ which looks like a regular neighbourhood of the desired minimal surface, such that ∂M has positive mean curvature; the condition of positive mean curvature is what is needed to apply one of Meeks and Yau's theorems to prove the existence of an embedded minimal surface Φ of the required type inside M, and since Φ lies inside M it will be of the form required to confirm the assertion of the bridge principle.

The execution of this programme by Meeks and Yau is complicated by various technical problems. For two reasons they allow M to have a piecewise-smooth boundary in a generalized sense, here called condition (C). The first is that it is easier to construct M if it is allowed to have a piecewise-smooth boundary. The second reason arises because we wish to construct minimal surfaces which may have positive genus, but the various forms of Dehn's lemma only apply to surfaces of genus zero. The surface that we wish to construct will be incompressible in M (in the sense of three-dimensional topology, see p.33), and under this strong topological assumption Meeks and Yau have proved the existence of an embedded minimal surface by a different method. Their theorem [MY2, Theorem 5] appears as our Theorem 4. The proof involves cutting a three-manifold along a minimal surface that may not be embedded and then treating one of the pieces as having a boundary with positive mean curvature. (An alternative approach to the existence of embedded incompressible minimal surfaces is due to Freedman, Hass and Scott [FHS], who give a detailed treatment only for surfaces without boundary but suggest that their methods should work for surfaces with boundary.)

On the other hand, to prove the existence of a minimal surface in M one proceeds by adding a collar to ∂M and giving the collar a metric that makes the union of M and the collar a complete manifold; for this M must be smooth and of strictly positive mean curvature. Furthermore, we wish to triangulate our minimal surface in order to undertake the cutting process in the proof of Theorem 4, and for this M must be real analytic. Therefore we need to approximate the original problem by one with better properties, and this is done in Chapter 1. The existence of a solution to the approximating problem is proved in Chapter 2, and in order to obtain a solution to the original problem we then need to show that the solutions to a sequence of approximating problems converge to a solution of the original problem, which is done in Chapter 3. The author encountered the difficulty that the convergence argument referred to by Meeks and Yau ([MY2, p.156] which refers to [MY1, pp.423-425]) assumes that the surfaces in question minimize area; in order to handle surfaces that do not necessarily minimize area, he has not been able to avoid making restrictions on curvature in Theorems 3 and 4, the purpose of these restrictions being that they allow the use of the isoperimetric inequality (3.11). The author does not know whether these restrictions on curvature can be removed.

Meeks and Yau do not give details of their convergence argument except for surfaces of the type of the disc. For surfaces other than discs, we have made use of the paper of Shiffman [Shiffman], which employs parallel-slit domains; there seems to be no other treatment in the literature that applies to surfaces of positive genus.

In Chapter 5 we construct examples which answer the two questions of Meeks mentioned above. Our constructions rely heavily upon the bridge principle. To perform these constructions rigorously we need to produce the "regular neighbourhood" M and then apply Theorem 4. In constructing M a key point is the ingenious technique due to Meeks and Yau by which a narrow tube may be added to a manifold with boundary of positive mean curvature so as to obtain a new manifold which still has boundary of positive mean curvature in the generalized sense. The reader will note that nowhere do we prove a general form of the bridge principle; the general form is hard to state, and it is more convenient to work with the bridge principle as a heuristic guide and then justify the applications individually.

CONVENTIONS

We shall frequently consider a manifold M with several Riemannian metrics on M. If g is a Riemannian metric on M we shall write

${}^{g}\Gamma^{k}_{ij}$ for the Christoffel symbols of g;

${}^{g}\nabla$ for the connection defined by g;

$d_g(a, b)$ for the distance between the points $a, b \in M$ measured with respect to g;

$L_g(c)$ for the length of a parametrised curve $c: [0, 1] \to M$ with respect to g;

$A_g(f)$ for the area of a parametrised surface $f: F \to M$ with respect to g, and

$A_g(F)$ for the area with respect to g of a surface F embedded in M;

$D_g(f) = \frac{1}{2}\int_F \|\nabla f\|^2$ for the energy or Dirichlet integral of a parametrised surface $f: F \to M$ with respect to g.

We shall omit the symbol g from these notations when it is clear which metric is in question.

If $f: F \to M$ is a parametrised surface in a three-manifold and ν is a choice of unit normal field on f, we define the mean curvature of f with respect to ν to be the function

$$h: F \to \mathbb{R}$$

given by

$$h(x) = \tfrac{1}{2}\,\mathrm{trace}(\langle S(X_\alpha, X_\beta), \nu\rangle),$$

where X_1, X_2 are an orthonormal basis for $T_x f$. With this definition the unit sphere S^2 in $\mathbb{R}^3$ has positive mean curvature with respect to the normal pointing towards its centre.

CHAPTER 1. APPROXIMATING A PIECEWISE-SMOOTH BOUNDARY BY A REAL ANALYTIC BOUNDARY.

Definition. A branched immersion of a Riemann surface F in a smooth three-manifold M [GOR, Definition 1.6] is a smooth map

$$f\colon F \to M,$$

where f is an immersion except at a discrete set of points, called branch points, and at each branch point there exist local coordinates (u^1, u^2) on F and (x^1, x^2, x^3) on M such that

$$\begin{aligned} f^1(w) + if^2(w) &= w^m + \sigma(w), \\ f^3(w) &= \chi(w), \end{aligned} \qquad (1.1)$$

$$\sigma(w) = o(|w|^m), \qquad \chi(w) = o(|w|^m),$$

$$\frac{\partial\sigma}{\partial u^i}(w) = o(|w|^{m-1}), \qquad \frac{\partial\chi}{\partial u^i}(w) = o(|w|^{m-1}), \qquad i = 1, 2,$$

where $w = u^1 + iu^2$, f^i are the components of f and m is an integer greater than 1.

<u>Definition</u>. A <u>minimal</u> <u>surface</u> in a smooth Riemannian three-manifold M is a continuous map

$$f: F \to M,$$

such that f intF is a harmonic branched immersion and conformal at immersed points.

A consequence of this definition is that the mean curvature of a minimal surface vanishes at immersed points, because in conformal coordinates the equation for vanishing mean curvature becomes the equation for a harmonic map.

The aim of this thesis is to construct certain minimal surfaces in Euclidean space, but we have to construct minimal surfaces in other manifolds as an intermediate stage. We shall show in Chapter 2 that it is possible to construct minimal surfaces in a three-manifold M if ∂M has strictly positive mean curvature with respect to the inward normal. In the rest of Chapter 1 we define a generalization of positive mean curvature due to Meeks and Yau [MY2], which we call condition (C), and prove that a three-manifold satisfying condition (C) can be approximated by manifolds with boundary of positive mean curvature. We shall use this process of approximation at two places in Chapter 4. First, the examples of minimal surfaces in Chapter 5 which are our goal are constructed so as to lie inside certain manifolds satisfying condition (C). Second, the proof that there exists an embedded minimal surface involves cutting a manifold along a minimal surface that is a candidate for being embedded and then proving the existence of a new minimal surface inside one of the pieces starting from the observation that it satisfies condition (C).

<u>Definition</u>. Let N be a C^2 Riemannian three-manifold and M be a compact three-dimensional submanifold of intN with piecewise-C^2 boundary ∂M. M is said to satisfy condition (C) if the following conditions are fulfilled:

> (C1) There is a C^2 Whitney stratification of N such that ∂M is a union of strata;
>
> (C2) each 2-dimensional stratum of ∂M has non-negative mean curvature with respect to the inward normal;
>
> (C3) each 2-dimensional stratum H of ∂M extends to a properly-embedded C^2 surface $\tilde{H}$ in N such that $H = \tilde{H} \cap M$;
>
> (C4) if closed 2-dimensional strata H_1 and H_2 meet at a point p, the inward normals to H_1 and H_2 are not at an angle of π at p.

Condition (C) is illustrated in Figure 1.

(a) Condition (C). (b) Not condition (C). (Not (C3))

Figure 1 (continued on page 17).

(c) Condition (C). (d) Not condition (C). (Not (C4))

Figure 1 (continued from page 16).

<u>Theorem</u> 1 (Meeks-Yau [MY2, proof of Theorem 1]). Let N be a compact real analytic three-dimensional manifold with a smooth metric g, M be a compact submanifold of intN satisfying condition (C) and G be a surface in ∂M bounded by a set Γ of rectifiable Jordan curves. Let $\varepsilon > 0$. Then there exists a real analytic metric g_ε on N such that g_ε and its derivatives of order less than $\frac{1}{\varepsilon}$ are within ε of g and its derivatives, a compact three-dimensional submanifold M_ε of intN with real analytic boundary ∂M_ε of strictly positive mean curvature and a set Γ_ε of disjoint real analytic Jordan curves in ∂M_ε such that there is an isotopy Φ from M to M_ε taking Γ to Γ_ε and moving each point of M by a g-distance of less than ε,

$$L_{g_\varepsilon}(\Gamma_\varepsilon) < L_g(\Gamma) + \varepsilon$$

and

$$A_{g_\varepsilon}(\Phi(G)) < A_g(G) + \varepsilon.$$

Proof. The first step is to perturb g to a real analytic metric g_ε in which each two-dimensional stratum of ∂M has strictly positive mean curvature. To construct g_ε we use a smooth function

$$\varphi: N \to \mathbb{R}$$

such that on each two-dimensional stratum H of ∂M

$$\nu . \nabla\varphi < 0,$$

where ν is the unit normal to H pointing into M. Take a finite set of open balls $\{B_k : k = 1, \ldots, \ell\}$ in N such that the B_k cover ∂M and, for each k, $\partial M \cap B_k$ is a disc. Define

$$\varphi_k : B_k \to \mathbb{R}$$

to satisfy

$$\nu . \nabla\varphi_k < 0$$

on each two-dimensional stratum H of ∂M that intersects B_k, by taking a diffeomorphism of B_k to a standard ball such that all the normals to $\partial M \cap B_k$ are close to a coordinate direction and pulling back the coordinate function in that direction. Then define φ by averaging the φ_k and extending by zero over the rest of N.

For $\delta > 0$, let $\gamma = \gamma_\delta$ be the metric on N given by

$$\gamma(p)(X, Y) = \exp(\delta\varphi(p))g(X, Y),$$

for all $p \in N$ and $X, Y \in T_pN$. Let H be a two-dimensional stratum of ∂M and ν be the unit normal to H pointing into M. We show that, in the metric γ, H has strictly positive mean curvature with respect to ν. If $p \in H$ and X_1, X_2 are vector fields defined on N in a neighbourhood of p such that $X_1(p)$, $X_2(p)$ are an orthonormal basis for T_pH, the mean curvature h of H at p with respect to ν in the metric γ is

$$h(p) = \tfrac{1}{2}\mathrm{trace}(\langle S(X_\alpha, X_\beta), \nu\rangle),$$

where the second fundamental form and inner product are taken with respect to γ. Now, writing $X = X_\alpha$, $Y = X_\beta$,

$$S(X, Y),$$

$$= \langle {}^\gamma\nabla_X Y, \nu\rangle$$

$$= \gamma_{k\ell} X^i\left(\frac{\partial Y^k}{\partial u^i} + {}^\gamma\Gamma^k_{ij} Y^j\right)\nu^\ell,$$

where u^i are normal coordinates for the metric g on N in a neighbourhood of p, and the suffices denote components in these coordinates,

$$= \gamma_{k\ell} X^i \Big(\frac{\partial Y^k}{\partial u^i} + (\exp(\delta\varphi)\,{}^g\Gamma^k_{ij} + \tfrac{1}{2}(g_{jm}\frac{\partial}{\partial u^i}\exp(\delta\varphi)$$

$$+ g_{im}\frac{\partial}{\partial u^j}\exp(\delta\varphi) - g_{ij}\frac{\partial}{\partial u^m}\exp(\delta\varphi))\gamma^{km})Y^j\Big)\nu^\ell .$$

Of the three terms involving the index m, the first two are zero since X and Y are orthogonal to ν. The third term is positive and does not vanish at p, so that since by our choice of coordinates the ${}^g\Gamma^k_{ij}$ do vanish at p, $\langle S(X, Y), \nu\rangle$ is positive at p.

By choosing a small enough δ, and taking g_ε to be a real analytic approximation to γ_δ (using the embedding theorem of Grauert [Grauert] in the arguments given by Narasimhan [Narasimhan]), we obtain g_ε with the properties in the statement of the theorem.

Because each two-dimensional stratum H is compact and has strictly positive mean curvature in the metric g_ε with respect to the inward normal ν, the mean curvature is still strictly positive over a neighbourhood of H in the surface $\tilde{H}$ given by condition (C3). Thus we may perturb the two-dimensional strata to make them transverse along each edge of intersection while maintaining condition (C).

We now show how to smooth an edge between two of the two-dimensional strata or two parts of the same stratum.

A one-dimensional stratum may be a line segment with distinct end points, a line segment with both end points the same or a circle.

First we consider the case of a line segment s with distinct end points that forms part of the intersection of the two-dimensional strata H_1 and H_2, which extend respectively to surfaces $\tilde{H}_1$ and $\tilde{H}_2$ properly embedded in N. Take a chart

$$\Psi: U \to N,$$

where U is open in $\mathbb{R}^3$ and $\Psi(U)$ is a neighbourhood of s, such that

$$\tilde{H}_1 \cap \Psi(U) = \Psi(\Pi_1 \cap U),$$

where Π_1 is the plane given by $x^1 = x^3$ in coordinates (x^1, x^2, x^3) on $\mathbb{R}^3$,

$$\tilde{H}_2 \cap \Psi(U) = \Psi(\Pi_2 \cap U),$$

where Π_2 is the plane given by $x^1 = -x^3$,

$$M \cap \Psi(U) \subset \Psi(P \cap U),$$

where $P = \{(x^1, x^2, x^3) \in \mathbb{R}^3 : x^3 \geqslant |x^1|\}$, and $\Psi^{-1}(s)$ is the interval $[0, 1]$ of the x^2-axis.

Then $\partial M \cap \Psi(U)$ lies in the image under Ψ of the graph of $x^3 = |x^1|$ as a function on the (x^1, x^2)-plane. We shall construct a new graph over the (x^1, x^2)-plane that will enable us to modify

∂M by replacing H_1 and H_2 with a single new stratum, in such a way that condition (C) is still satisfied. Let

$$f\colon \mathbb{R} \to \mathbb{R}$$

be the C^2 function such that

$$f'''(x) = \begin{cases} 0 \text{ for } x < -1, \\ 2 \text{ for } -1 < x < 0, \\ -2 \text{ for } 0 < x < 1, \\ 0 \text{ for } x > 1, \end{cases}$$

$$f''(x) = 0 \text{ for } |x| > 1,$$

$$f'(x) = \begin{cases} -1 \text{ for } x < -1, \\ 1 \text{ for } x > 1, \end{cases}$$

$$f(0) = \tfrac{1}{3}.$$

The graphs of f and its derivatives are illustrated in Figure 2.

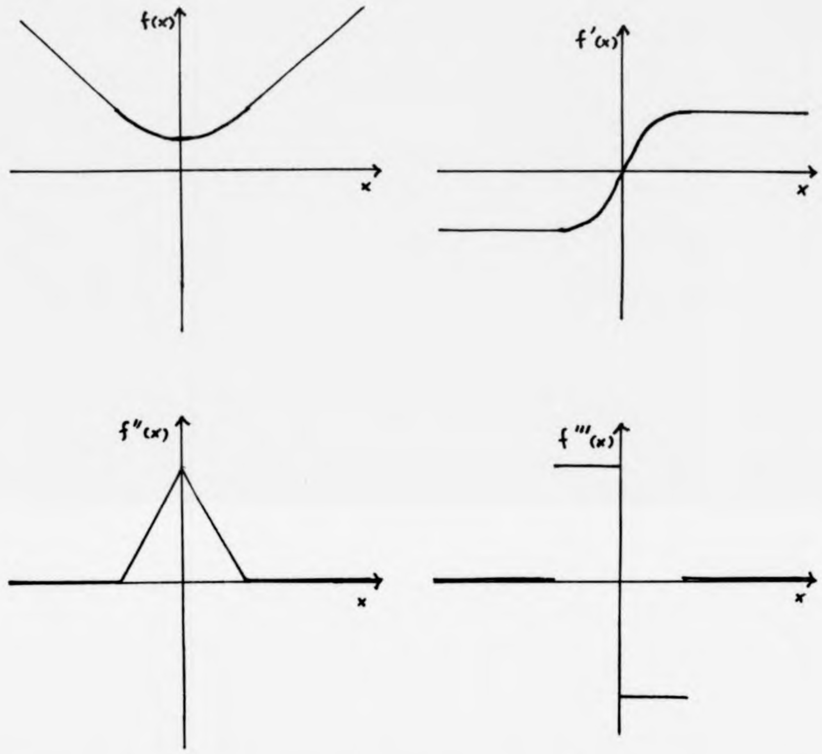

Figure 2. The auxiliary function f and its derivatives.

$f(x) = |x|$ for $|x| > 1$, so that if for $\eta > 0$ we define

$$F_\eta(x^1, x^2) = \eta f\left(\frac{x^1}{\eta}\right)$$

the graph of $x^3 = F_\eta(x^1, x^2)$ coincides with the graph of $x^3 = |x^1|$ for $|x^1| > \eta$. Now we show that, when η is small enough, on a compact subset of the intersection of U with the (x^1, x^2)-plane which contains the η-neighbourhood of $\Psi^{-1}(s)$, the graph of $x^3 = F_\eta(x^1, x^2)$ lies inside U and has positive mean curvature with respect to the upward normal in the metric $\Psi^*(g_\varepsilon)$. Define new coordinates (y^1, y^2, y^3) on $\mathbb{R}^3$ by

$$y^1 = \frac{x^1}{\eta}, \qquad y^2 = x^2, \qquad y^3 = \frac{x^3}{\eta},$$

and let

$$\sigma_\eta : \mathbb{R}^3 \to \mathbb{R}^3$$

be the map that sends (y^1, y^2, y^3) to (x^1, x^2, x^3). Then

$$\Psi_\eta = \Psi \circ (\sigma_\eta | \sigma_\eta^{-1}(U))$$

is a chart on N with the same image as Ψ and

$$\sigma_\eta^{-1}(\{(x^1, x^2, x^3) : x^3 = F_\eta(x^1, x^2)\})$$

$$= \{(y^1, y^2, y^3) : y^3 = f(y^1)\},$$

which is the same surface S for all values of η. At points in the domain of Ψ_η with $|y^1| > 1$, S coincides with $\Psi_\eta^{-1}(H_1)$ or $\Psi_\eta^{-1}(H_2)$ and consequently has positive mean curvature with respect to the upward normal in the metric $\gamma = \Psi_\eta^*(g_\varepsilon)$. At the point $p = (y^1, y^2, f(y^1))$ of S, let X be the unit vector parallel to $(1, \frac{-\gamma_{21} + \gamma_{23}f'}{\gamma_{22}}, f')$, Y be the unit vector parallel to (0, 1, 0) and ν be the unit upward normal; then X and Y form an orthonormal basis for T_pS. We consider the function h of (y^1, y^2) which is the mean curvature with respect to ν at the corresponding point p of S. We can extend X, Y, ν to vector fields on the domain of Ψ_η which are independent of y^3 and obtain the formula

$$h(y^1, y^2) = \tfrac{1}{2}\gamma_{ij}\left(X^k\left(\frac{\partial X^i}{\partial y^k} + {}^{\gamma}\Gamma^i_{k\ell}X^\ell\right) + Y^k\left(\frac{\partial Y^i}{\partial y^k} + {}^{\gamma}\Gamma^i_{k\ell}Y^\ell\right)\right)\nu^j. \quad (1.2)$$

Let $\eta_0 > 0$ be a choice of η such that the set

$$A = \{(y^1, y^2, y^3) : |y^1| \leqslant 1,\ 0 \leqslant y^2 \leqslant 1,\ |y^3| \leqslant 1\}$$

lies in the domain of Ψ_{η_0}. Then for $\eta = \eta_0$, the components of γ are bounded and bounded away from 0 and their derivatives are bounded on A.

We now claim that there exists $0 < \zeta < 1$ such that $h > 0$ for $\zeta \leqslant |y^1| \leqslant 1$ and $0 \leqslant y^2 \leqslant 1$. Consider the expression for $\frac{\partial h}{\partial y^1}$ on $0 < |y^1| < 1$ obtained by differentiating (1.2). As η tends to zero, all terms remain bounded except for the derivative of the term in $\frac{\partial X^i}{\partial y^1}$. This gives rise to a term in f''' and terms in f'', together with terms which are bounded as η tends to zero. The terms in f'' are small when $|y^1|$ is close to 1, whereas the term in f''' tends to positive infinity uniformly in $0 < |y^1| < 1$ as η tends to zero. Therefore there exists $0 < \zeta < 1$ such that, for sufficiently small η, $h' > 0$ for $-1 < y^1 < -\zeta$, $0 \leqslant y^2 \leqslant 1$ and $h' < 0$ for $\zeta < y^1 < 1$,

$0 \leqslant y^2 \leqslant 1$. Since h is continuous and $h > 0$ for $|y^1| = 1$ it follows that, for sufficiently small η, $h > 0$ for $\tau \leqslant |y^1| \leqslant 1$, $0 \leqslant y^2 \leqslant 1$.

For $|y^1| \leqslant \tau$, $0 \leqslant y^2 \leqslant 1$, all terms in (1.2) remain bounded as η tends to zero except for the term in $\frac{\partial x^i}{\partial y^1}$ which tends to positive infinity. Together with the claim established in the last paragraph, this shows that, for sufficiently small η, $h > 0$ for $|y^1| \leqslant 1$, $0 \leqslant y^2 \leqslant 1$.

So far we have considered the case where s is a line segment with distinct end points. If s is a line segment with both end points the same then the argument is the same except that Ψ is not an embedding. If s is a circle we take two charts Ψ_1 and Ψ_2 whose images cover s and average the functions F_η in these charts so as to obtain a C^2 function with the required properties.

The result of this smoothing process is a new manifold M' satisfying condition (C) such that $\partial M'$ has one less edge than ∂M. By applying the process inductively to the edges of ∂M we obtain a C^2 surface G with strictly positive mean curvature. By choosing sufficiently small values of η we may make G as close as we wish to ∂M. We may then choose ∂M_ε to be a real analytic approximation to G and Γ_ε to be a set of real analytic curves on ∂M_ε close to Γ. $\square$

CHAPTER 2. AN EXISTENCE THEOREM FOR MINIMAL SURFACES.

The theorem of this chapter is stated in the proof of Theorem 5 of Meeks and Yau [MY2]. For the details of the construction of (P, h) we have referred to [MY1, p.412]. To prove that the surface we obtain lies in M we have replaced their argument [MY2, pp.155-156] by a more geometrical one, which, however, relies upon a differentiability condition.

Lemma 1 (Meeks-Yau [MY2, proof of Theorem 2]). Let M be a smooth three-dimensional Riemannian manifold. Let

$$f\colon B \to M,$$

where B is the unit disc in $\mathbb{R}^2$, be a minimal immersion and let

$$g\colon B \to M$$

be an immersion with non-negative mean curvature with respect to the normal ν. Suppose that f and g are tangent at 0 and that no point of f lies on the opposite side of g from ν. Then there are neighbourhoods U, V of 0 such that

$$f(U) = g(V).$$

Proof. Take coordinates (x^1, x^2, x^3) on M near f(0) such that (0, 0, 0) corresponds to f(0), the common tangent plane to f and g at (0, 0, 0) is the (x^1, x^2)-plane and ν points in the positive x^3-direction. In a neighbourhood of 0, f is a graph $x^3 = \varphi(x^1, x^2)$ and g is a graph $x^3 = \chi(x^1, x^2)$. φ satisfies the minimal surface equation (1.2) which after multiplication by $1 + \gamma^{ij}\varphi_i\varphi_j$ becomes

$$\sum_{ij} A_{ij}(\nabla\varphi)\varphi_{ij} + B(\nabla\varphi) = 0, \tag{2.1}$$

where $\nabla\varphi = (\varphi_1, \varphi_2)$ and A_{ij}, B are polynomials with coefficients which are functions of position in M. χ satisfies the inequality that its mean curvature is non-negative with respect to ν, and so satisfies the inequality

$$\sum_{ij} A_{ij}(\nabla\chi)\chi_{ij} + B(\nabla\chi) \geq 0. \tag{2.2}$$

Subtract (2.1) from (2.2) to obtain

$$\sum_{ij} (A_{ij}(\nabla\chi)(\chi_{ij} - \varphi_{ij}) + (A_{ij}(\nabla\chi) - A_{ij}(\nabla\varphi))\varphi_{ij})$$

$$+ B(\nabla\chi) - B(\nabla\varphi) \geq 0.$$

Each of $(A_{ij}(\nabla\chi) - A_{ij}(\nabla\varphi))$, $(B(\nabla\chi) - B(\nabla\varphi))$ is a sum of terms of the form

$$\theta(x^1, x^2, \chi(x^1, x^2))\chi_1^a\chi_2^b - \theta(x^1, x^2, \varphi(x^1, x^2))\varphi_1^a\varphi_2^b,$$

where θ is a smooth function and $a, b \in \mathbb{N}$,

$$= \sum_{\alpha=0}^{a} \theta(x^1, x^2, \chi(x^1, x^2))\chi_1^{a-\alpha-1}\varphi_1^{\alpha}\chi_2^{b}(\chi_1 - \varphi_1)$$

$$+ \sum_{\beta=0}^{b} \theta(x^1, x^2, \chi(x^1, x^2))\varphi_1^{a}\chi_2^{b-\beta-1}\varphi_2^{\beta}(\chi_2 - \varphi_2)$$

$$+ (\theta(x^1, x^2, \chi(x^1, x^2)) - \theta(x^1, x^2, \varphi(x^1, x^2)))\varphi_1^{a}\varphi_2^{b}. \quad (2.3)$$

Let

$$\Theta(x^1, x^2): [0, 1] \to \mathbb{R}$$

be the smooth function defined by

$$\Theta(x^1, x^2)(s) = \theta(x^1, x^2, \varphi(x^1, x^2) + s(\chi(x^1, x^2) - \varphi(x^1, x^2))).$$

We have

$$\theta(x^1, x^2, \chi(x^1, x^2)) - \theta(x^1, x^2, \varphi(x^1, x^2))$$

$$= \int_0^1 \frac{d\Theta}{ds}(x^1, x^2)\, ds$$

$$= (\chi(x^1, x^2) - \varphi(x^1, x^2)) \int_{\varphi(x^1, x^2)}^{\chi(x^1, x^2)} \frac{\partial\theta}{\partial x^3}\, dx .$$

After substituting this expression in the last term of (2.3), we may substitute the values of the derivatives of φ and χ in (2.3) to conclude that $\Psi = \chi - \varphi$ satisfies a second order elliptic linear homogeneous inequality with smooth coefficients

$$L\Psi \geq 0.$$

Ψ is non-positive because no point of f lies on the opposite side of g from ν. On a small disc centred at the origin L is uniformly elliptic and we may apply the Hopf maximum principle [GT, Theorem 3.5] to conclude that Ψ is identically zero. $\square$

In the proof of Theorem 2 we shall apply a theorem of Morrey [Morrey, Theorem 9.4.8] which asserts the existence of a minimal surface. To apply this theorem in a case where the surface need not be of genus zero, we need a space of Riemann surfaces which is suitable for discussing convergence of conformal structures.

Definition [Shiffman, p.857]. A normalized slit domain F consists of the upper half-plane $\{z \in \mathbb{C}: \mathrm{Im}(z) \geqslant 0\}$ in which are distinguished as slits some finite number of disjoint line segments of the form

$$\{z \in \mathbb{C}: \mathrm{Re}(z) \geqslant \alpha,\ \mathrm{Im}(z) = \beta > 0\},$$

called infinite slits, or

$$\{z \in \mathbb{C}: \alpha_1 \leqslant \mathrm{Re}(z) \leqslant \alpha_2,\ \mathrm{Im}(z) = \beta > 0\},$$

called finite slits. Each side of each slit is divided into finitely many segments, called edges, as is the real axis; each of the edges is either regarded as part of the boundary or glued to another edge by identifying points with the same real part; in the second case the identified edges are regarded as lying in the interior of F.

After the identifications on these edges the remaining edges become a union of circles, and when we refer to the <u>boundary</u> ∂F of F it is this union of circles that we shall understand. It is further required that there should exist α such that all points on edges with real part greater than α are identified, except possibly for points on the real axis. The end points of the edges will be called the <u>vertices</u> of F. The domains are required to be normalized by the condition that either

(2.4.1) the infinite slit which extends most to the left has $\alpha = 0$ and the topmost infinite slit has $\beta = 1$, or

(2.4.2) there are no infinite slits and no identifications and the centre of one of the slits is at i.

We shall make use of the concept of a cohesive minimizing sequence introduced by Courant [Courant, p.145].

<u>Definition</u>. Let

$$f_n : F_n \to E, \qquad n \in \mathbb{N},$$

be a sequence of continuous maps from Riemann surfaces F_n to a metric space E. If there exists $\varepsilon > 0$ such that, for all closed curves $\gamma : S^1 \to F_n$ and all n,

$$\operatorname{diam} \operatorname{im}(f_n \circ \gamma) < \varepsilon \Rightarrow \gamma \text{ is null-homotopic},$$

then $\{f_n\}$ is said to be <u>cohesive</u>.

Shiffman proves that if $f_n: F_n \to E$ is such a cohesive sequence and the domains F_n are normalized slit domains of the same topological type then a subsequence of $\{F_n\}$ tends to a limit F which is a slit domain of the same topological type as F_n.

Supposing that the sequence $\{F_n\}$ converges to F, we wish to discuss the convergence of the functions f_n. For this purpose we regard a normalized slit domain before the identifications on the edges as a subset of the complex plane; thus it is meaningful to speak of "a disc of radius r." Shiffman proves that we may assume that all the F_n have the same arrangement of edges and identifications, though not necessarily the same as that of F [Shiffman, p.861]. If $x \in \mathrm{int}F$ does not lie on a slit, there exists $r > 0$ such that $B(x, r)$ intersects none of the slits in F_n for large enough n; to discuss the convergence of $\{f_n\}$ at x, we discuss convergence on $B(x, r)$.

If $x \in \mathrm{int}F$ lies on an edge then in general the arrangement of edges in $B(x, r)$ will depend on n. Since the identifications in F_n and F respect the metric on $\mathbb{C}$ they induce metrics d_{F_n} on F_n and d_F on F. We define

$$B_H(x, r) = \{y \in H: d_H(y, x) < r\},$$

where H is F_n or F. Choose $r > 0$ such that there is an arc c in

$\partial B(x, r)$ which for large enough n does not intersect any slits.
Let

$$r_n: B_{F_n}(x, r) \to B_F(x, r) \tag{2.5}$$

be the unique conformal diffeomorphism that fixes x and the end points of c. To discuss the convergence of $\{f_n\}$ at x, we discuss the convergence of $\{f_n \circ r_n\}$ on $B_F(x, r)$, which we identify with a disc of radius r in $\mathbb{C}$. (In the case wnere x does not lie on a slit, we can take r_n to be the identity.)

If $x \in \partial F$ then x lies on at least one edge E. Take edges E_n in F_n converging to E and apply a translation parallel to the imaginary axis to make E_n coincide with E; then by a similar device we may discuss convergence in a half-disc neighbourhood of x.

Definition [Hempel, p.58]. Let M be a three-manifold and let G be a compact connected surface, not a disc or a sphere, which is either properly embedded in M or contained in ∂M. We say that G is incompressible in M if for every disc D embedded in M with $D \cap G = \partial D$, ∂D is null-homotopic in G.

Theorem 2 (Meeks-Yau [MY2]). Let N be a three-dimensional smooth Riemannian manifold without boundary and M be a compact three-dimensional submanifold with smooth boundary ∂M that has strictly positive mean curvature with respect to the inward normal. Let Γ

be a set of Jordan curves in ∂M that bounds a compact connected subdomain G of ∂M which is either a disc or incompressible in M. Then there is a Riemann surface F homeomorphic to G and a minimal surface $f: F \to M$ such that $f(\partial F) = \Gamma$ and f is a monotone continuous map of degree 1 on each component of ∂F, satisfying $A(f) \leq A(G)$. If $\pi_2(M) = 0$, f is homotopic relative to Γ to a map onto G that is a homeomorphism on intF.

<u>Proof</u>. Consider the parallel surfaces Φ_ε to ∂M

$$\Phi_\varepsilon = \{\exp_p \varepsilon n(p) : p \in \partial M\},$$

where n(p) is the outward unit normal to ∂M at p. There exists a $\rho > 0$ such that for $0 \leq \varepsilon \leq \rho$, Φ_ε is defined and is a smooth embedded surface of strictly positive mean curvature with respect to the normal pointing into M.

We define the auxiliary function

$$\beta : \{x \in \mathbb{R} : x < \rho\} \to \mathbb{R}$$

by

$$\beta(x) = \begin{cases} \dfrac{(\exp(-\frac{1}{\rho}))^2}{(\exp(-\frac{1}{\rho}) - \exp(-\frac{1}{x}))^2} & 0 < x < \rho, \\ 1 & x \leq 0. \end{cases}$$

Now we define P to be the manifold $\{p \in N: d_g(p, M) < \rho\}$ with the metric

$$h_{ij}(p) = \beta(d_g(p, M))g_{ij}(p), \tag{2.6}$$

where g is the metric of N.

Our construction of β makes h a complete smooth metric. We claim that h is <u>homogeneously regular</u>, which means that there exist positive constants c and C such that every point p of P is the image of 0 in a chart φ whose domain is the unit ball B(0, 1), satisfying

$$c|v|^2 \leq h_{ij}(\varphi(x))v^i v^j \leq C|v|^2 \tag{2.7}$$

for all $x \in B(0, 1)$, $v \in T_x B(0, 1)$.

Let $\{B_k: k = 1, \ldots, \ell\}$ be a finite set of open geodesic balls in N that covers P. For each k take a chart $\Psi_k: U_k \to B_k$ where U_k is open in $\mathbb{R}^3$ and Ψ_k extends to $\overline{U}_k$. Then there exist positive constants c_0 and C_0 such that

$$c_0|v|^2 \leq g_{ij}(\Psi_k(x))v^i v^j \leq C_0|v|^2$$

for all $k = 1, \ldots, \ell$, $x \in U_k$, $v \in T_x U_k$. By composing with a change of scale we may assume that $C_0 \leq 1$.

(P, h) is homogeneously regular if the complement of a compact subset is homogeneously regular. Let $\delta_0 > 0$ be a number such that for any $p \in P$ there exists k such that

$$\bar{B}(\Psi_k^{-1}(p),\ \delta_0) \subset U_k .$$

Then for any $p \in P$ with $\delta = d_g(p, \Phi_p) < 2\delta_0$ there exists k such that

$$\bar{B}(\Psi_k^{-1}(p),\ \tfrac{\delta}{2}) \subset U_k .$$

Since $C_0 \leq 1$,

$$\Psi_k(\bar{B}(\Psi_k^{-1}(p),\ \tfrac{\delta}{2})) \subset P.$$

The configuration is illustrated in Figure 3.

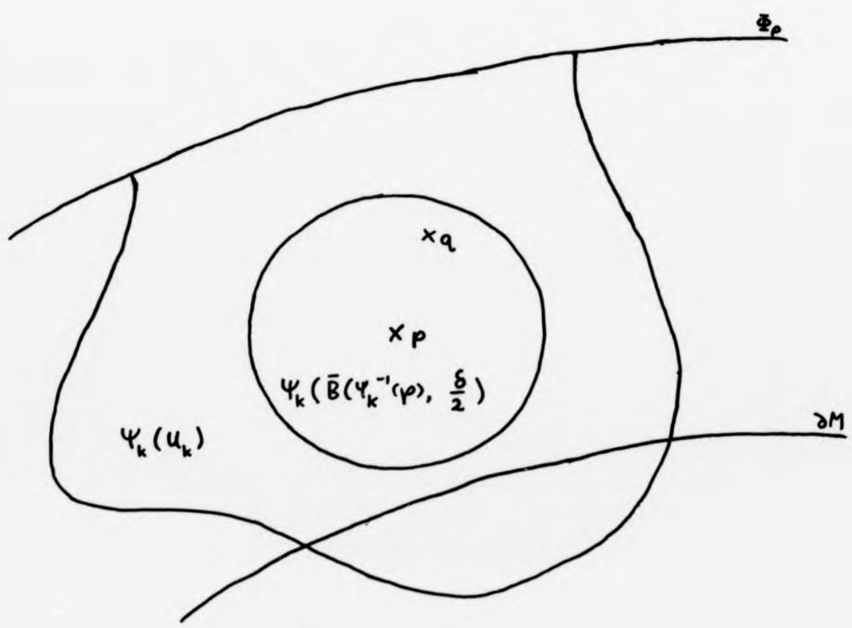

Figure 3.

By composing $\Psi_k | B(\Psi_k^{-1}(p), \frac{\delta}{2})$ with a translation of $\mathbb{R}^3$ we obtain a chart

$$\Psi_p : B(0, \frac{\delta}{2}) \to P$$

such that $\Psi_p(0) = p$, satisfying

$$c_0 |v|^2 \leq g_{ij}(\Psi_k(p)) v^i v^j \leq C_0 |v|^2.$$

If σ: $B(0, 1) \to B(0, \frac{\delta}{2})$ is the linear radial contraction, then $\varphi_p = \Psi_p \circ \sigma$ is a chart whose domain is $B(0, 1)$ satisfying

$$c_0 |v|^2 \leq \frac{4}{\delta^2} g_{ij}(\varphi_p(x)) v^i v^j \leq C_0 |v|^2. \tag{2.8}$$

Now

$$\beta(x) = \rho^4 (\rho - x)^{-2} + O((\rho - x)^{-1})$$

and so there exist $\eta > 0$, $c_1 > 0$, $C_1 > 0$ such that for $\rho - \eta < x < \rho$

$$c_1 (\rho - x)^{-2} \leq \beta(x) \leq C_1 (\rho - x)^{-2}. \tag{2.9}$$

Since $C_0 \leq 1$, for q in the range of φ_p

$$\frac{\delta}{2} \leq d_g(q, \Phi_\rho) \leq \frac{3\delta}{2},$$

where $\delta = d_g(p, \Phi_\rho)$, and in a neighbourhood of Φ_ρ we have also

$$d_g(q, \Phi_\rho) = \rho - d_g(q, M).$$

Therefore by combining (2.6), (2.8) and (2.9) we obtain (2.7) in a neighbourhood of Φ_ρ, with

$$c = \frac{c_0 c_1}{9}, \qquad C = C_0 C_1.$$

We consider the class $\mathcal{F}$ of continuous maps $f\colon F \to P$ such that

(2.10.1) F is a normalized slit domain of the topological type of G,

(2.10.2) f takes ∂F homeomorphically onto Γ,

(2.10.3) f is homotopic to a homeomorphism from F to G, relative to ∂F.

If G is not a disc, it is isotopic to a two-sided properly embedded incompressible surface in P, and so the inclusion of G in P induces a monomorphism of fundamental groups [Hempel, Corollary 6.2]. Because P is homogeneously regular, any subset of diameter less than the constant $2c$ lies in the image of a chart the domain of which is a ball. Thus if $f \in \mathcal{F}$ and $\gamma\colon S^1 \to F$ is a closed curve satisfying

$$\operatorname{diam} \operatorname{im}(f \circ \gamma) < 2c,$$

$f \circ \gamma$ is null-homotopic in P and so γ is null-homotopic in F. Therefore any sequence in $\mathcal{F}$ is cohesive. This fact may be used in the proof of Morrey's existence theorem [Morrey, Theorem 9.4.8] to show that there exists a minimal branched immersion $f: F \to P$ which minimizes area among surfaces in $\mathcal{F}$.

A theorem of Osserman, as proved by Gulliver for a Riemannian manifold [Gulliver, Theorem 8.1], states that f has no true branch point; in other words, the image of f is an immersed submanifold.

We now show that f(intF) lies in intM. If not, let $x \in$ intF be such that $f(x) \notin$ intM and $d_h(f(x), M)$ achieves the maximum among points with this property. Let Φ_r be the surface parallel to ∂M which passes through f(x). Then at f(x) the immersed submanifold im(f) is tangent to Φ_r, because otherwise x would not achieve the maximum distance from M. Since Φ_r has positive mean curvature with respect to the inward normal we derive a contradiction from Lemma 1.

Now that we have established that f(intF) $\subset$ intM, it follows that f is an immersion by a theorem of Gulliver, Osserman and Royden on branched immersions [GOR, Theorem 6.3]. $\square$

CHAPTER 3. THE CONVERGENCE OF A SEQUENCE OF MINIMAL SURFACES IN PERTURBED METRICS.

The tneorem of this chapter enables us to assert that the solutions of the minimal surface equation found in Chapter 2 for a convergent sequence of perturbed metrics as constructed in Chapter 1 have a subsequence which converges to a solution for the original metric. Our proof is based on Meeks and Yau [MY1, Theorem 2]. In order to carry out their proof for surfaces which may not minimize area, the author has found it necessary to impose the curvature bound (3.9.5).

Lemma 2 ([MY1, Appendix 1] This lemma is similar to a growth estimate of Siu and Yau [SY, Proposition 1.10].). Let M be a Riemannian manifold of dimension m whose sectional curvature is bounded from above by $K = \varkappa^2 > 0$. Let N be a minimal submanifold of M of dimension n such that for some $x \in N$ and $\varepsilon > 0$, $B(x, \varepsilon)$ does not meet ∂M or ∂N, where $B(x, \varepsilon)$ denotes the open ball in M of centre x and radius ε. Let $0 < \delta < \min(\varepsilon, i(M))$, where $i(M)$ is the injectivity radius of M. Then

$$A(N \cap B(x, \delta)) \geq \Xi \varkappa^{-n} \int_0^{\delta} t^{-1} (\sin(\varkappa t))^n \, dt,$$

where $\Xi > 0$ depends only on n.

Proof. Let r be distance from x measured in M and Δ be the Laplacian of N. Our first step is to estimate Δr^2 using a comparison theorem.

If X, Y $\in T_pM$, extend X and Y to vector fields $\tilde{X}$, $\tilde{Y}$ and define the Hessian of M to be the quadratic form

$$H(f)(X, Y) = (\tilde{X}\tilde{Y}f)(p) - ((\nabla_{\tilde{X}}\tilde{Y})f)(p).$$

Let M_0 be the Euclidean m-sphere of radius $\frac{1}{K}$;

$$\gamma : [0, a] \to M,$$

$$\gamma_0 : [0, a] \to M_0$$

be geodesics parametrized by arc length with $\gamma(0) = x$, $a < i(M)$; r_0 be distance from $\gamma_0(a)$; and $X \in T_{\gamma(a)}M$, $X_0 \in T_{\gamma_0(a)}M$, $|X| = |X_0| = 1$,

$$\langle X, \frac{\partial}{\partial r}\rangle = \langle X_0, \frac{\partial}{\partial r_0}\rangle = 0.$$

Then a comparison theorem given by Siu and Yau [SY, p.227] states that

$$H(r)(X, X) \geq H(r_0)(X_0, X_0).$$

Therefore

$$\begin{aligned} & H(r^2)(X, X) \\ = \;& 2aH(r)(X, X) \\ \geq \;& 2aH(r_0)(X_0, X_0) \\ = \;& 2\varkappa a \cot(\varkappa a) \end{aligned} \tag{3.1}$$

Now for any $Y \in T_x M$ write

$$Y = Y' + \lambda \frac{\partial}{\partial r},$$

where $\langle Y', \frac{\partial}{\partial r} \rangle = 0$. Then

$$\begin{aligned} & (\nabla_{Y'} \frac{\partial}{\partial r}) r \\ = \;& \langle \nabla_{Y'} \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle \\ = \;& \tfrac{1}{2} Y' \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle \\ = \;& 0 \end{aligned}$$

and therefore

$$H(r^2)(Y, Y) = H(r^2)(Y', Y') + 2\lambda^2.$$

It follows from (3.1) with $X = \frac{Y'}{|Y'|}$ that

$$H(r^2)(Y, Y) \geqslant 2\kappa a \cot(\kappa a). \tag{3.2}$$

Let $y \in N$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $T_y N$ such that

$$\langle e_i, \frac{\partial}{\partial r}\rangle = 0, \qquad i = 2, \ldots, n.$$

Extend e_i to a vector field $\tilde{e}_i$. Then

$$\sum_{i=1}^{n} H(r^2)(e_i, e_i)$$

$$= \sum_{i=1}^{n} ((\tilde{e}_i \tilde{e}_i r^2)(y) - ((\nabla_{\tilde{e}_i} \tilde{e}_i)^T r^2)(y) - ((\nabla_{\tilde{e}_i} \tilde{e}_i)^\perp r^2)(y))$$

$$= \Delta r^2 \tag{3.3}$$

because N is minimal and so the third term vanishes. Combining (3.2) and (3.3) we have

$$\Delta r^2 \geqslant 2n\kappa r \cot(\kappa r) \tag{3.4}$$

for $r < \min(\varepsilon, i(M))$. By Sard's theorem, for almost all $0 < t < \min(\varepsilon, i(M))$, $\partial(N \cap B(x, t))$ is a smooth manifold. Since $|\nabla r| \leqslant 1$, where ∇ is the gradient measured in N, for such t

$$t\, A(\partial(N \cap B(x, t)) \geqslant \int_{\partial(N \cap B(x, t))} r \nabla r . \nu ,$$

where ν is the unit outward normal to $\partial(N \cap B(x, t))$ in N,

$$= \frac{1}{2} \int_{N \cap B(x,\ t)} \Delta r^2$$

by the divergence theorem,

$$\geq n\kappa \int_{N \cap B(x,\ t)} r \cot(\kappa r) \qquad (3.5)$$

by (3.4). Let

$$C(t) \quad = \quad \int_{N \cap B(x,\ t)} r \cot(\kappa r)$$

$$= \int_{r\,=\,0}^{t} \left(\int_{\partial(N \cap B(x,\ r))} r \cot(\kappa r) \frac{1}{|\nabla r|} \right) dr.$$

We proceed to estimate C. Since $|\nabla r| \leq 1$, we have

$$\frac{dC}{dt}(t) \geq t \cot(\kappa t) \ A(\partial(N \cap B(x,\ t))$$

$$\geq n\kappa \cot(\kappa t) \ C(t) \qquad (3.6)$$

by (3.5). The function $(\sin(\kappa t))^n$ satisfies

$$\frac{d}{dt}(\sin(\kappa t))^n = n\kappa \cot(\kappa t)(\sin(\kappa t))^n. \qquad (3.7)$$

Let

$$\lim_{t \to 0} C(t) \ (\sin(\kappa t))^{-n} = \kappa^{-n-1} \Omega \ ;$$

then Ω depends only on n. By integrating (3.6) and (3.7) we obtain

$$C(t) \geq \kappa^{-n-1}\Omega(\sin(\kappa t))^{n} \tag{3.8}$$

for $0 < t < \min(\varepsilon, i(M))$. Now, since $|\nabla r| \leq 1$,

$$A(N \cap B(x, \delta)) \geq \int_0^{\delta} A(\partial(N \cap B(x, t)))\, dt$$

$$\geq n\kappa \int_0^{\delta} t^{-1} C(t)\, dt \qquad \text{by (3.5)}$$

$$\geq \Xi \kappa^{-n} \int_0^{\delta} t^{-1} (\sin(\kappa t))^{n}\, dt$$

by (3.8), where $\Xi = n\Omega$. $\square$

<u>Lemma</u> 3 ([Courant, Lemma 3.1]). Let $a > 0$. Let U be open in $\mathbb{C}$ and

$$h: U \to M$$

be a smooth map into a Riemannian manifold such that

$$D(h) \leq a.$$

Let x be any point of $\mathbb{C}$ and C_r be the subset of the circle of centre x and radius r which lies in U. Given $0 < \delta < 1$, there exists $\delta \leq \rho \leq \sqrt{\delta}$ such that either C_ρ is empty or

$$(L(h(C_\rho)))^2 \leqslant \frac{8\pi a}{\log(\frac{1}{\delta})}.$$

Proof. If

$$p(r) = r\int_{C_r} \left\|\frac{dh}{ds}\right\|^2 ds,$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm,

$$\int_\delta^{\sqrt{\delta}} \frac{p(r)}{r}\,dr = \int_\delta^{\sqrt{\delta}} \left\|\frac{dh}{ds}\right\|^2 ds\,dr \leqslant 2D(h) \leqslant 2a$$

and so by the mean value theorem of the integral calculus there exists $\delta \leq \rho \leq \sqrt{\delta}$ such that

$$\int_{C_\rho} \left\|\frac{dh}{ds}\right\|^2 ds \leqslant \frac{4a}{\log(\frac{1}{\delta})}\frac{1}{\rho}.$$

The result follows by the Schwarz inequality. $\square$

Theorem 3 (Meeks-Yau [MY1, Theorem 2]). Let N be a compact smooth three-dimensional manifold with boundary and $\{g^n\}$ a sequence of smooth Riemannian metrics on N that tend in the smooth topology to a smooth metric g. Let $K > 0$ be an upper bound for the sectional curvatures of the g^n and g. Let $\{M_n\}$ be a sequence of compact subsets of N that tend in g to a compact subset M. Let $\{\Gamma_n\}$ be a sequence of sets of disjoint rectifiable Jordan curves in M_n that tend in g to a set Γ of disjoint rectifiable Jordan curves in M: suppose that

$$L_g(\Gamma_n) \to L_g(\Gamma).$$

Let

$$f_n : F_n \to N$$

be a sequence of maps such that

(3.9.1) F_n is a normalized slit domain of the same topological type for all n,

(3.9.2) $f_n | \mathrm{int} F_n$ is a conformal harmonic the metric g^n,

(3.9.3) ∂F has the same number of components as Γ, $f(\partial F) = \Gamma$ and f is a monotone continuous map of degree 1 on each component of ∂F,

(3.9.4) $f_n(F_n) \subset M_n$,

(3.9.5) $A_{g^n}(f_n) \leqslant a$, where a is a constant independent of n satisfying $0 < a < \frac{4\pi}{K}$,

(3.9.6) $\{f_n\}$ is a cohesive sequence,

(3.9.7) if the F_n are discs, there exist distinct points $p_1, p_2, p_3 \in \partial F$, distinct points $q_1, q_2, q_3 \in \Gamma$ and conformal diffeomorphisms

$$r_n : F \to F_n$$

such that

$$f_n \circ r_n(p_i) \to q_i, \qquad i = 1, 2, 3.$$

Then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a map

$$f: F \to N$$

such that

(3.10.1) F is a normalized slit domain which is the non-degenerate limit of the F_{n_k}, and in particular has the same topological type,

(3.10.2) $f|\mathrm{int}F$ is a conformal harmonic immersion for the metric g,

(3.10.3) $f(\partial F) = \Gamma$ and f is a monotone continuous map of degree 1 on each component of ∂F,

(3.10.4) $f(F) \subset M$,

(3.10.5) $A_g(f) \leqslant a$,

(3.10.6) if r_n is the map defined by (2.5), $\{f_{n_k} \circ r_{n_k}\}$ converges to f in the smooth topology.

Proof. Since condition (3.9.6) of the theorem states that $\{f_n\}$ is a cohesive sequence, by taking a subsequence we may assume the F_n tend to a limit F. We shall discuss the convergence of the f_n at points of F by using the functions r_n of (2.5) and writing $\tilde{f}_n = f_n \circ r_n$.

First we show that a subsequence of $\{f_n\}$ converges on ∂F [Courant, Lemma 3.2]. Because Γ_n tends to Γ and $L_g(\Gamma_n)$ tends to $L_g(\Gamma)$, given any $\sigma > 0$ there exists $\tau(\sigma) > 0$ such that if α and β are points of the same component Δ of Γ_n or Γ and $d_g(\alpha, \beta) \leqslant \tau$ then the g-length of one of the arcs of Δ determined by α and β does not exceed σ. Because $g^n \to g$ uniformly the sequence $\{f_n\}$ has bounded Dirichlet integral with respect to g. Therefore by Lemma 3 there exists $\delta > 0$ such that for all n and all $x \in \partial F$ there exists $\delta \leqslant \rho \leqslant \sqrt{\delta}$ such that

$$L(\tilde{f}_n(C_\rho)) \leqslant \tau,$$

where C_ρ is the arc of the circle of centre x and radius ρ that lies in the domain of $\tilde{f}_n$. In particular if y and z are the end points of C_ρ then

$$d(x, y) \leqslant \delta, \qquad d(x, z) \leqslant \delta$$

and

$$d_g(f_n(y), f_n(z)) \leqslant \tau.$$

We claim that the image under f_n of the shorter arc of ∂F joining y to z is the shorter arc of Γ_n joining $f_n(y)$ to $f_n(z)$; this implies that $\{f_n\}$ is equicontinuous at x. If the F_n are discs we take the functions r_n as in condition (3.9.7); if we choose δ so that at most one of the p_i is within δ of x, it follows that the image under $\tilde{f}_n$ of the shorter arc joining y to z is the shorter arc of Γ_n joining $\tilde{f}_n(y)$ to $\tilde{f}_n(z)$. Suppose now that the F_n are not discs; let τ be the longer arc of ∂F joining y to z and suppose that $f_n(\tau)$ is the shorter arc of Γ_n joining $f_n(y)$ to $f_n(z)$. Then $\tau \cup C_\rho$ is an essential closed curve in F but

$$\mathrm{diam}(f_n(\tau \cup C_\rho)) < \sigma,$$

which can be chosen arbitrarily small; this contradicts condition (3.9.6) which states that $\{f_n\}$ is a cohesive sequence. Therefore $\{f_n\}$ is equicontinuous on ∂F and it follows by the Arzelà-Ascoli theorem that a subsequence converges on ∂F.

We now prove that a subsequence of $\{f_n\}$ converges in a neighbourhood of $x \in \mathrm{int}F$. The metrics g^n and g are all uniformly equivalent, that is there exists $C > 0$ such that

$$\frac{1}{C}\, g'_{ij}(x)v^i v^j \leq g''_{ij}(x)v^i v^j \leq C\, g'_{ij}(x)v^i v^j,$$

where $x \in N$, $v \in T_xN$ and g', g'' are any elements of $\{g^n, g\}$. Therefore there exists $\rho > 0$ such that a geodesic ball of the metric g of radius less than or equal to ρ is an embedded ball which is strictly convex with respect to all of the metrics g^n, g. Let $\{B_k : k = 1, \ldots, \ell\}$

be a cover of N by such balls. There exists $\varepsilon > 0$ such that if B is a geodesic ball for some g^n of radius less than ε then $B \subset B_k$ for some k and the g^n-distance from B to ∂B_k is greater than ε.

Since the Dirichlet integrals of $\{\tilde{f}_n\}$ have an upper bound by (3.9.5) we may apply Lemma 3 to show that if $0 < \delta < 1$ is small enough for $B(x, \delta)$ to lie in the domain of $\tilde{f}_n$ then there exists $\delta \leq \rho_n \leq \sqrt{\delta}$ such that

$$(L_{g^n}(\tilde{f}_n(\partial B(x, \rho_n))))^2 \leq \frac{8\pi a}{\log(\frac{1}{\delta})}.$$

If δ is sufficiently small then

$$\frac{8\pi a}{\log(\frac{1}{\delta})} < \varepsilon^2,$$

so that

$$\tilde{f}_n(\partial B(x, \rho_n)) \subset B_{k_n}$$

for some k_n and the g^n-distance of $\tilde{f}_n(\partial B(x, \rho_n))$ from ∂B_{k_n} is greater than ε.

We claim that, for small enough δ,

$$\tilde{f}_n(B(x, \rho_n)) \subset B_{k_n}$$

for all n. By the Theorema Egregium, the upper bound K on the sectional curvature of g^n implies that the Gaussian curvature of the minimal surface f_n is also bounded above by K. Under this condition the following isoperimetric inequality holds for a simply-connected domain $D \subset F_n$ [Osserman, Theorem 4.3][Toponogov, Theorem 1]. (The presence of branch points may be handled by a perturbation of the metric.)

$$(L(f_n|\partial D))^2 \geqslant 4\pi A(f_n|D) - K(A(f_n|D))^2$$

$$\geqslant (4\pi - aK)A(f_n|D), \tag{3.11}$$

where a is the number satisfying $4\pi - aK > 0$ given by condition (3.9.5). Therefore, given $\eta > 0$, we may choose $\delta > 0$ such that for all n

$$A_{g^n}(\tilde{f}_n|B(x, \rho_n)) < \eta. \tag{3.12}$$

Assume that for some n

$$\tilde{f}_n(B(x, \rho_n)) \not\subset B_{k_n}.$$

Then for that n we can find

$$y \in \tilde{f}_n(B(x, \rho_n)) \cap \partial B_{k_n},$$

so that

$$d(y, \tilde{f}_n(\partial B(x, \rho_n)) > \varepsilon.$$

By Lemma 2

$$A(\mathrm{im}(\tilde{f}_n) \cap B(y, \varepsilon)) \geqslant \Xi \varkappa^{-n} \int_0^{\varepsilon} t^{-1}(\sin(\varkappa t))^n \, dt,$$

where $K = \varkappa^2$, and so if we choose

$$\eta = \Xi \varkappa^{-n} \int_0^{\varepsilon} t^{-1}(\sin(\varkappa t))^n \, dt$$

we derive a contradiction to (3.12).

We can now prove that $\{f_n\}$ is equicontinuous on compact subsets of int$F$. As $\delta \to 0$,

$$\mathrm{diam}(\tilde{f}_n(\partial B(x, \rho_n))) \to 0.$$

We claim that also, as $\delta \to 0$,

$$\mathrm{diam}(\tilde{f}_n(B(x, \rho_n))) \to 0.$$

Let $\Delta(\delta, n)$ be a geodesic ball of the metric g containing $\tilde{f}_n(\partial B(x, \rho_n))$ such that

$$\mathrm{diam}(\Delta(\delta, n)) = \mathrm{diam}(\tilde{f}_n(\partial B(x, \rho_n))).$$

Foliate $\overline{B}_k \setminus \mathrm{int}\Delta(\delta, n)$ with geodesic spheres Σ_λ of the metric g, depending continuously on $\lambda \in [0, 1]$, $\Sigma_0 = \partial\Delta(\delta, n)$, $\Sigma_1 = \partial B_k$, all of which are geodesically convex in all metrics g^n, g because their g-radius is less than ρ. If

$$\tilde{f}_n(B(x, \rho_n)) \not\subset \Delta(\delta, n),$$

let λ_0 be the largest λ such that Σ_λ intersects $\tilde{f}_n(B(x, \rho_n))$. Apply Lemma 1 to Σ_{λ_0} and $\tilde{f}_n$ to derive a contradiction. This shows that $\{f_n\}$ is equicontinuous at x.

We now adapt the preceding argument to prove equicontinuity in a neighbourhood of a boundary point. In this case $\{\tilde{f}_n\}$ is a sequence of functions on $B(x, R) \cap H$, where H is a half-plane through x. Given $\varepsilon > 0$, by choosing $r > 0$ and applying Lemma 3 we can find $\rho_n > 0$ such that for all n

$$L_{g^n}(\tilde{f}_n | (\partial B(x, \rho_n) \cap H)) < \frac{\varepsilon}{2}.$$

Since the boundary values of $\{f_n\}$ are equicontinuous we can also choose r so that

$$L_{g^n}(\tilde{f}_n | (B(x, \rho_n) \cap \partial H)) < \frac{\varepsilon}{2}.$$

We prove equicontinuity as before by applying the isoperimetric inequality (3.11) to the domain $B(x, \rho_n) \cap H$.

We have now proved that $\{f_n\}$ is equicontinuous on the compact set F. It follows by the Arzelà-Ascoli theorem that a subsequence of $\{f_n\}$ converges on F to a continuous function $f\colon F \to M$.

We now show that a subsequence of $\{f_n\}$ converges in the smooth topology on compact subsets of intF. By taking a subsequence we may assume that there exists k independent of n such that

$$f_n(B(x, \delta)) \subset B_k$$

for some δ and all n. Let γ be the metric on B_k obtained by pulling back the Euclidean metric over a chart with image B_k; then, since the metrics g^n are all uniformly equivalent, the uniform bound (3.9.5) for the Dirichlet integrals $D_{g^n}(f_n)$ implies a uniform bound for the Euclidean Dirichlet integrals $D_\gamma(\tilde{f}_n | B(x, \delta))$. We shall use this to derive a uniform bound for $|\nabla \tilde{f}_n|$.

In a coordinate chart for N with image B_k, the condition that $\tilde{f}_n$ is harmonic in the metric g^n becomes

$$\begin{aligned} \Delta \tilde{f}^i_n &= \frac{\partial^2 \tilde{f}^i_n}{\partial \xi^2} + \frac{\partial^2 \tilde{f}^i_n}{\partial \eta^2} \\ &= -\sum_{jk} {}^n\Gamma^i_{jk}\left\{\frac{\partial \tilde{f}^j_n}{\partial \xi}\frac{\partial \tilde{f}^k_n}{\partial \xi} + \frac{\partial \tilde{f}^j_n}{\partial \eta}\frac{\partial \tilde{f}^k_n}{\partial \eta}\right\} \end{aligned} \tag{3.13}$$

for all i, where (ξ, η) are coordinates on $B(x, \delta)$, $\tilde{f}^i_n$ is the i'th component of $\tilde{f}_n$, Δ is the Euclidean Laplacian and ${}^n\Gamma^i_{jk}$ are the

Christoffel symbols of g^n. Let φ be a smooth function with compact support in $B(x, \delta)$, equal to 1 on $B(x, \frac{\delta}{2})$. Then

$$\Delta(\varphi\tilde{f}^i_n) = \varphi\Delta\tilde{f}^i_n + 2\nabla\varphi.\nabla\tilde{f}^i_n + \tilde{f}^i_n\Delta\varphi. \tag{3.14}$$

From (3.13)

$$\varphi\Delta\tilde{f}^i_n = -\varphi\sum_{jk}\Gamma^i_{jk}\left\{\frac{\partial\tilde{f}^j_n}{\partial\xi}\frac{\partial\tilde{f}^k_n}{\partial\xi} + \frac{\partial\tilde{f}^j_n}{\partial\eta}\frac{\partial\tilde{f}^k_n}{\partial\eta}\right\}$$

$$= -\sum_{jk}{}^n\Gamma^i_{jk}\left\{\left(\frac{\partial(\varphi\tilde{f}^j_n)}{\partial\xi} - \tilde{f}^j_n\frac{\partial\varphi}{\partial\xi}\right)\frac{\partial\tilde{f}^k_n}{\partial\xi} + \left(\frac{\partial(\varphi\tilde{f}^j_n)}{\partial\eta} - \tilde{f}^j_n\frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\tilde{f}^k_n}{\partial\eta}\right\}. \tag{3.15}$$

Since the functions $\tilde{f}^i_n$, φ, $\frac{\partial\varphi}{\partial\xi}$, $\frac{\partial\varphi}{\partial\eta}$, $\Delta\varphi$ and ${}^n\Gamma^i_{jk}$ are uniformly bounded on $B(x, \delta)$, (3.14) and (3.15) imply the inequality

$$|\Delta(\varphi\tilde{f}_n)| \leq c_1|\nabla(\varphi\tilde{f}_n)||\nabla\tilde{f}_n| + c_2|\nabla\tilde{f}_n| + c_3 \tag{3.16}$$

for some constants c_1, c_2, c_3 depending on φ but not on n. Taking the $L^{\frac{4}{3}}$ norm of (3.16) we obtain

$$\left(\int_{B(x,\ \delta)} |\Delta(\varphi\tilde{f}_n)|^{\frac{4}{3}}\right)^{\frac{3}{4}}$$

$$\leq c_4\left(\int_{B(x,\ \delta)} |\nabla(\varphi\tilde{f}_n)|^{\frac{4}{3}}|\nabla\tilde{f}_n|^{\frac{4}{3}}\right)^{\frac{3}{4}} + c_5\left(\int_{B(x,\ \delta)} |\nabla\tilde{f}_n|^{\frac{4}{3}}\right)^{\frac{3}{4}} + c_6$$

for some constants c_4, c_5, c_6,

$$\leqslant c_4 \left(\int_{B(x,\ \delta)} |\nabla \tilde{f}_n|^2 \right)^{\frac{1}{2}} \left(\int_{B(x,\ \delta)} |\nabla(\varphi \tilde{f}_n)|^4 \right)^{\frac{1}{4}}$$

$$+ c_7 \left(\int_{B(x,\ \delta)} |\nabla f_n|^2 \right)^{\frac{1}{2}} + c_6, \tag{3.17}$$

after estimating the first term by Hölder's inequality and the second by the inequality of the norms.

Because $\varphi \tilde{f}_n$ has compact support, we have Sobolev's inequality

$$\left(\int_{B(x,\ \delta)} |\nabla(\varphi \tilde{f}_n)|^4 \right)^{\frac{1}{4}} \leqslant c_8 \left(\int_{B(x,\ \delta)} |\nabla^2(\varphi \tilde{f}_n)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \tag{3.18}$$

for some constant c_8 [GT, Theorem 7.10] and an $L^{\frac{4}{3}}$ estimate which implies

$$\left(\int_{B(x,\ \delta)} |\nabla^2(\varphi \tilde{f}_n)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leqslant c_9 \left(\int_{B(x,\ \delta)} |\Delta(\varphi \tilde{f}_n)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \tag{3.19}$$

for some constant c_9. (As usually stated, the L^p estimate involves another term on the right-hand side, but this need not appear for a function of compact support. In Agmon's derivation of the L^p estimate [Agmon, Theorem 6.1] one may proceed by using his inequality (5.11) instead of inequality (5.12) in the proof of Lemma 5.1.) We combine (3.17), (3.18) and (3.19) to obtain

$$\left(1 - c_4 c_8 c_9 \left(\left(\int_{B(x,\,\delta)} |\nabla \tilde{f}_n|^2\right)^{\frac{1}{2}}\right)\right)\left(\left(\int_{B(x,\,\delta)} |\nabla^2(\varphi \tilde{f}_n)|^{\frac{4}{3}}\right)^{\frac{3}{4}}\right.$$

$$\leq c_7 c_9 \left(\left(\int_{B(x,\,\delta)} |\nabla \tilde{f}_n|^2\right)^{\frac{1}{2}} + c_6 c_9 \qquad (3.20)$$

We have seen that there is a uniform estimate for the integrals

$$D_\gamma(\tilde{f}_n | B(x,\,\delta)) = \tfrac{1}{2}\int_{B(x,\,\delta)} |\nabla \tilde{f}_n|^2.$$

Therefore (3.20) gives a uniform estimate for $\int_{B(x,\,\delta)} |\nabla^2(\varphi \tilde{f}_n)|^{\frac{4}{3}}$.

The Sobolev inequality (3.18) gives a uniform estimate for $\int_{B(x,\,\delta)} |\nabla(\varphi \tilde{f}_n)|^4$, and hence for $\int_{B(x,\,\frac{\delta}{2})} |\nabla \tilde{f}_n|^4$, since $\varphi = 1$ on $B(x, \frac{\delta}{2})$. We now take χ to be a smooth function with compact support in $B(x, \frac{\delta}{2})$, equal to 1 on $B(x, \frac{\delta}{3})$. By the same method as for (3.16) we derive the inequality

$$|\Delta(\chi\tilde{f}_n)| \leq c_{10}|\nabla(\chi\tilde{f}_n)|\,|\nabla\tilde{f}_n| + c_{11}|\nabla\tilde{f}_n| + c_{12},$$

for some constants c_{10}, c_{11}, c_{12}. This gives a uniform estimate for $\int_{B(x, \frac{\delta}{2})} |\Delta(\chi\tilde{f}_n)|^2$ and hence, by the L^2 analogue of (3.19), a uniform estimate for $\int_{B(x, \frac{\delta}{2})} |\nabla^2(\chi\tilde{f}_n)|^2$. By Sobolev's inequality this gives a uniform estimate for $\int_{B(x, \frac{\delta}{2})} |\nabla(\chi\tilde{f}_n)|^6$. We repeat this argument on $B(x, \frac{\delta}{3})$ to obtain a uniform estimate for $\int_{B(x, \frac{\delta}{3})} |\nabla^2(\psi\tilde{f}_n)|^3$, where ψ is some function with compact support in $B(x, \frac{\delta}{3})$, equal to 1 on $B(x, \frac{\delta}{4})$. Sobolev's inequality then gives a uniform estimate for $|\nabla\tilde{f}_n|$ on $B(x, \frac{\delta}{4})$.

Therefore the right-hand side of (3.13) is bounded uniformly in n. A theorem of Morrey and Nirenberg [GT, Theorem 11.4], together with our estimate for $|\nabla\tilde{f}_n|$, gives a uniform $C^{1+\alpha}$ estimate for $\tilde{f}_n$, for some $0 < \alpha < 1$. Now the right-hand side of (3.13) is C^{α}, and by iterating the Schauder estimates we obtain uniform $C^{i+\alpha}$ estimates of $\tilde{f}_n$ for all i.

Therefore a subsequence of $\{f_n\}$ converges in the smooth topology on compact subsets of intF. It follows that the limit f is smooth, and moreover conformal and harmonic in the metric g. $f(M) \subset M$ because f is the limit of a subsequence of $\{f_n\}$. f is monotone on each component of ∂F because f is the limit of a subsequence of $\{f_n\}$. $\square$

CHAPTER 4. A THEOREM ON THE EXISTENCE OF EMBEDDED MINIMAL SURFACES.

In this chapter we prove a theorem of Meeks and Yau that asserts the existence of embedded minimal surfaces in certain three-manifolds [MY2, Theorem 5]. The theorem has a weaker conclusion than the geometric Dehn lemma also due to Meeks and Yau [MY1, Theorem 5] in that it does not assert that the embedded surface minimizes area, but the hypotheses are also weaker in that they do not restrict the applications to surfaces of genus zero. The theorem is preceded by two lemmas on unique continuation.

Lemma 4 (Meeks-Yau [MY1, Lemma 2]). Let M be a smooth three-dimensional Riemannian manifold and let

$$f_i : B \to M, \qquad i = 1, 2,$$

where B is the unit disc in $\mathbb{C}$, be minimal immersions such that

$$f_1(0) = f_2(0).$$

Then there are neighbourhoods U_1 and U_2 of 0 such that either

$$f_1(U_1) = f_2(U_2)$$

or $f_1(U_1)$ intersects $f_2(U_2)$ along a finite number of curves passing through $f_1(0)$ and the intersection is transverse at points other than $f_1(0)$.

Proof. If the intersection is transverse at $f_1(0)$ the conclusion follows immediately. Otherwise take coordinates (x^1, x^2, x^3) on M near $f_1(0)$ such that (0, 0, 0) corresponds to $f_1(0)$ and the common tangent plane to f_1 and f_2 at (0, 0, 0) is the (x^1, x^2)-plane. In a neighbourhood of 0, f_1 is a graph $x^3 = \varphi_1(x^1, x^2)$ and f_2 is a graph $x^3 = \varphi_2(x^1, x^2)$. Because φ_1 and φ_2 satisfy the minimal surface equation we may argue as in the proof of Lemma 1 that $\varphi = \varphi_1 - \varphi_2$ satisfies a second order elliptic linear homogeneous equation with smooth coefficients

$$L\varphi = 0.$$

If φ vanishes to infinite order at 0, the unique continuation theorem of Aronszajn [Aronszajn] and Cordes [Cordes] shows that φ is zero on a neighbourhood of 0, so that there exist neighbourhoods U_1 and U_2 of 0 such that

$$f_1(U_1) = f_2(U_2).$$

If φ vanishes to order N, Bers [Bers, paragraph 4] has given an asymptotic expansion

$$\varphi(x) = p_N(x) + O(|x|^{N-1+\varepsilon}),$$

and

$$\nabla\varphi(x) = \nabla p_N(x) + 0(|x|^{N-2+\varepsilon}),$$

where $x = (x^1, x^2)$, $0 < \varepsilon < 1$ and $p_N(x)$ is a homogeneous polynomial of degree N satisfying the second order elliptic linear homogeneous equation with constant coefficients tangential at 0 to $L\varphi = 0$. Cheng [Cheng, Theorem 2.2] has applied a theorem of Kuo [Kuo, Theorem 1] to prove that in a neighbourhood of 0

$$\varphi(x) = p_N(u(x)),$$

where $u: \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 diffeomorphism. Therefore in a neighbourhood of 0 which contains no other critical points of $p_N \circ u$ the graphs of φ_1 and φ_2 intersect transversely except at 0 and their intersection consists of a finite number of curves. $\square$

<u>Lemma</u> 5 (Meeks-Yau [MY1, Lemma 5]). Let M be a smooth three-dimensional Riemannian manifold and let

$$f_i: B \cap H \to M, \qquad i = 1, 2,$$

where B is the unit disc in $\mathbb{C}$ and H is the upper half-plane $\{z: \mathrm{Im}(z) \geq 0\}$, be minimal immersions such that

$$f_1(0) = f_2(0),$$

$$f_1(I) = f_2(I),$$

where I is the interval $(-1, 1)$ of the real axis, and, at each point of $f_1(I)$, f_1 is tangent to f_2. Then there are points $\xi_i \in I$, $i = 1, 2$, and neighbourhoods U_i of ξ_i in $H \cap B$ such that

$$f_1(U_1) = f_2(U_2).$$

Proof. By a theorem of Nitsche [Nitsche3, Theorem 2] f_1 and f_2 have finitely many branch points on I. Let $p = f_1(\xi_1) = f_2(\xi_2)$ be none of these, and take coordinates (x^1, x^2, x^3) on M near p such that part of the x^1-axis corresponds to $f_1(I)$, $(0, 0, 0)$ corresponds to p, the common tangent plane to f_1 and f_2 at $(0, 0, 0)$ is the (x^1, x^2)-plane and, for some $\rho > 0$, f_1 is locally a graph $x^3 = \varphi_1(x^1, x^2)$ over $B_\rho(0) \cap H$ and f_2 is locally a graph $x^3 = \varphi_2(x^1, x^2)$ over $B_\rho(0) \cap H$. For $|x^1| < \rho$ and $n \geq 0$ we have

$$\frac{\partial^n \varphi_1}{(\partial x^1)^n}((x^1, 0)) = \frac{\partial^n \varphi_2}{(\partial x^1)^n}((x^1, 0)) = 0$$

by choice of coordinates. Since f_1 and f_2 are tangent along $f_1(I)$ we have

$$\frac{\partial^{n+1} \varphi_1}{(\partial x^1)^n \partial x^2}((x^1, 0)) = \frac{\partial^{n+1} \varphi_2}{(\partial x^1)^n \partial x^2}((x^1, 0)) = 0$$

for $|x^1| < \rho$ and $n \geq 0$. Because φ_1 and φ_2 satisfy the minimal surface equation we may argue as in the proof of Lemma 1 that $\varphi = \varphi_1 - \varphi_2$ satisfies a second order elliptic linear homogeneous equation with smooth coefficients

$$L\varphi = 0. \tag{4.1}$$

We see from this equation that $\frac{\partial^2 \varphi}{(\partial x^2)^2}((x', 0))$ vanishes for $|x'| < \rho$ and consequently so do all its x'-derivatives. By differentiating the equation (4.1) with respect to x' we may now show that all partial derivatives of φ vanish on $\{(x', 0): |x'| < \rho\}$. The unique continuation theorem of Aronszajn [Aronszajn] and Cordes [Cordes] shows that φ vanishes on a neighbourhood of (0, 0), which implies the conclusion of the lemma. □

<u>Theorem</u> 4 (Meeks-Yau [MY2, Theorem 5]). Let N be a smooth Riemannian three-manifold and M be a compact three-dimensional submanifold which satisfies condition (C); suppose further that for each two-dimensional stratum H of ∂M the surface $\tilde{H}$ to which it extends by (C3) has non-negative mean curvature on a neighbourhood of H. Let G be a compact connected subdomain of ∂M with non-empty boundary consisting of rectifiable curves such that G is incompressible in M. Suppose that the sectional curvatures of M are bounded above by a constant K such that

$$K\, A(G) < 4\pi.$$

Then there is a Riemann surface F homeomorphic to G and a minimal surface

$$f \colon F \to M$$

such that $f(\partial F) = \partial G$,

f induces a monotone continuous map of degree 1 on each component of ∂F; and f(F) is embedded. If $\pi_2(M) = 0$, f(F) is isotopic to G, relative to ∂G.

Proof. Because the outline of this proof is obscured by the technical detail, we begin by giving a simplified sketch. We consider the set $\mathcal{G}$ of minimal surfaces in M with boundary ∂G and area not greater than A(G) which are homotopic relative to the boundary to a homeomorphism onto G, and let

$$f \colon F \to M$$

be an element of $\mathcal{G}$ which is as close as possible to G. f cuts M into a finite number of pieces; we let P be the piece which contains G and show that we may regard P as a manifold satisfying condition (C). If the image of f is not embedded, part of it lies outside P.

Now we find a minimal surface

$$h\colon H \to P$$

with boundary ∂G and $A(h) \leq A(G)$, homotopic relative to ∂H to a homeomorpnism onto G. From the fact that h cannot be closer to G than f one can prove that h is the same as f. Therefore the image of f lies in P, and we have derived a contradiction which completes our sketch proof.

We now give the proof in detail. For each n we consider the real analytic manifold $M_{\frac{1}{n}}$ approximating M and the set of real analytic Jordan curves $\Gamma_{\frac{1}{n}}$ approximating $\Gamma = \partial G$ given by Theorem 1. Let G_n be the domain of $\partial M_{\frac{1}{n}}$ that corresponds to G. There exists a constant $K' \geq K$ and an integer n_0 such that for all $n \geq n_0$ the sectional curvatures of $M_{\frac{1}{n}}$ are bounded above by K' and

$$K' \sup_{n \geq n_0} A(G\ ) < 4\pi.$$

Let $\mathcal{G}$ be the set of minimal surfaces

$$f\colon F \to M_{\frac{1}{n}} \tag{4.2.1}$$

such that

(4.2.2) F is a normalized slit domain homeomorphic to G,

(4.2.3) $f(\mathrm{int}F) \subset \mathrm{int}M_{\frac{1}{n}}$,

(4.2.4) f takes ∂F homeomorphically onto $\Gamma_{\frac{1}{n}}$,

(4.2.5) f is homotopic relative to ∂F to a homeomorphism from F to G_n, assuming $\pi_2(M) = 0$,

(4.2.6) $A(f) \leqslant A(G_n)$.

Since an analytic minimal surface is a homeomorphism on the boundary [HH], Theorem 2 shows that $\mathcal{G}$ is not empty. Theorem 3, with all metrics equal to $g_{\frac{1}{n}}$, all manifolds equal to $M_{\frac{1}{n}}$ and all sets of Jordan curves equal to $\Gamma_{\frac{1}{n}}$, together with the lower semicontinuity of the area functional, shows that, when $n \geqslant n_0$, $\mathcal{G}$ is sequentially compact. For $f \in \mathcal{G}$ we now define a number C(f) that measures the <u>closeness</u> of f to G : let

$$C(f) = \int_{\Gamma_{\frac{1}{n}}} \nu . \xi ,$$

where ν is the unit inward normal of $\partial M_{\frac{1}{n}}$ and ξ is the unit normal of f pointing away from G_n. Because the curves of $\Gamma_{\frac{1}{n}}$ are analytic f can be analytically continued across the boundary by a theorem of Hildebrandt [HH] and so the integral C exists. Because $\mathcal{G}$ is sequentially compact we may find an element of $\mathcal{G}$

$$\varphi : \Phi \to M$$

such that

$$C(\varphi) = \sup_{f \in \mathcal{G}} C(f).$$

We claim that φ is an embedding on a neighbourhood of $\partial\Phi$. Otherwise there would be a sequence of self-intersections tending to some point $x \in \partial\Phi$ at which φ would therefore have a branch point. Hildebrandt's result allows us to derive the expansion (1.1) at x [GL]. Now the existence of self-intersections shows that, in the expansion (1.1), $\arg(\varphi^1(w) + i\varphi^2(w))$ must take all values for points w which lie in Φ. Since $\partial M_{\frac{1}{n}}$ has strictly positive mean curvature, this implies that there are points of $\mathrm{im}(\varphi)$ which lie outside M , contradicting property (4.2.1) of the functions in $\mathcal{G}$. Since the extended φ sends a smooth arc in $\partial\Phi$ containing x to a smooth arc in $\Gamma_{\frac{1}{n}}$ we conclude that φ is an immersion at x.

Suppose now that φ is an immersion.

Let Π be the closure of the component of $M_{\frac{1}{n}} \setminus \mathrm{im}(\varphi)$ whose closure contains G_n. We shall modify Π to obtain a manifold P satisfying condition (C) such that $G_n \subset P$. Because $\mathrm{im}(\varphi)$ is a real analytic set it may be triangulated [Lojasiewicz], so that the frontier $\dot{\Pi}$ of Π in N is the union of a finite number of smooth surfaces. Lemma 4 shows that points where two of these surfaces intersect non-transversely are isolated; furthermore, if $\varphi(D_1)$ and $\varphi(D_2)$, for D_1 and D_2 discs in Φ, are pieces of surface that intersect non-transversely at $x \in N$, near x their intersection is transverse and consists of 2ν curves, where $\nu \geqslant 2$; thus, for small $\rho > 0$, $B_\rho(x) \setminus \varphi(D_1 \cup D_2)$ consists of $2\nu + 2$ components, $A_1, \ldots, A_{2\nu+2}$, 2 of which, say A_1 and A_2, are distinguished by containing points on the exponential image of one or other normal to $\varphi(D_1)$ at x. If Π meets A_1 or A_2 then Π is disjoint from $A_3, \ldots, A_{2\nu+2}$ and any component H of $\mathring{\Pi} \cap \varphi(D_1 \cup D_2)$ may be extended to a surface $\tilde{\tilde{H}}$ as in condition (C). If Π does not meet A_1 or A_2 then Π may meet one or more of $A_3, \ldots, A_{2\nu+2}$; the surfaces in $\mathring{\Pi}$ do not satisfy (C4) at x and $\mathring{\Pi}$ may fail to be a topologically embedded surface. To avoid these problems we make Π slightly smaller: in each A_ℓ, $\ell = 3, \ldots, 2\nu + 2$, that intersects Π we introduce an additional boundary surface Σ_ℓ cutting off a portion of A_ℓ near x; Σ_ℓ is required to be transverse to $\varphi(D_1)$ and $\varphi(D_2)$ and to have non-negative mean curvature with respect to the normal pointing away from x. By modifying Π in this way near each point of non-transverse self-intersection of φ we obtain a manifold P satisfying condition (C) such that $G_n \subset \lambda P$.

If φ is not an immersion, Π does not satisfy condition (C) near a branch point, and so a further modification is needed. Let $x \in \operatorname{int} \Phi$ be a branch point of φ and take a chart $\chi\colon U \to V$, where U is open in $\mathbb{R}^3$ and V is a neighbourhood of $\varphi(x)$. Then $\chi^{-1} \circ \varphi | W\colon W \to \mathbb{R}^3$ defines a branched immersion on some neighbourhood W of x. Let R_0 be a sphere of large radius tangent to $\chi^{-1} \circ \varphi(W)$ at $\chi^{-1} \circ \varphi(x)$; we suppose that the inward normal to $\chi^{-1}(\Pi)$ at x is an outward normal to R_0. Now let R_1 be a sphere of the same centre as R_0 and slightly larger radius, so that a cap Q of R_1 near $\chi^{-1} \circ \varphi(x)$

intersects Π in a small disc. Let $P = \chi(Q)$. Then P has the right intersection with $\dot{\Pi}$ to be a surface in the stratification, but does not give condition (C) because it has negative mean curvature with respect to the inward normal. However, we may make the mean curvature of P arbitrarily small by choosing the radius of R_0 to be large. The first stage of the approximation procedure of Chapter I is a change in the metric of M (p. 19), and if the mean curvature of P is small enough in the old metric the mean curvature of P in the new metric will be positive. Thus with a suitable choice of P for each k we may obtain an approximating sequence of real analytic manifolds $\{P_{\frac{1}{k}}\}$ by applying Theorem I.

For each k let $P_{\frac{1}{k}}$ be the real analytic manifold approximating P given by Theorem 1 and let G_k' be the domain of $\partial P_{\frac{1}{k}}$ that corresponds to G_n. By Theorem 2 there exists for each k a minimal surface

$$\chi_k : X_k \rightarrow P_{\frac{1}{k}} \tag{4.3.1}$$

such that

(4.3.2) X_k is a normalized slit domain homeomorphic to G, (4.3.3) $\chi_k(\mathrm{int}X_k) \subset \mathrm{int}P_{\frac{1}{k}}$,

(4.3.4) χ_k takes ∂X_k homeomorphically onto $\partial G_k'$,

(4.3.5) χ_k is homotopic relative to ∂X_k to a homeomorphism from X_k to G_k', assuming $\pi_2(M) = 0$,

(4.3.6) $A(\chi_k) \leq A(G_k')$.

There exists a constant $K'' \geq K'$ and an integer k_0 such that for all $k \geq k_0$ the sectional curvatures of $P_{\frac{1}{k}}$ are bounded above by K'' and

$$K'' \sup_{k \geq k_0} A(G_k') < 4\pi.$$

By Theorem 3 a subsequence of $\{\chi_k\}$ converges to a minimal surface

$$\chi : X \rightarrow P$$

which satisfies conditions (4.2) if we regard it as a map into $M_{\frac{1}{n}}$. The unit normal μ to χ pointing away from G_n must be equal to ξ, otherwise φ would not maximize C. Lemma 5 and Lemma 4 now imply that the images of χ and φ coincide.

Since $\varphi(\mathrm{int}\Phi) \subset \mathrm{int}M_{\frac{1}{n}}$, if φ is not an embedding there are two points $x, y \in \mathrm{int}\Phi$ with

$$\varphi(x) = \varphi(y).$$

By Lemma 4 there are neighbourhoods U and V of x and y such that either

$$\varphi(U) = \varphi(V)$$

or $\varphi(U)$ intersects $\varphi(V)$ transversely except at $\varphi(x)$.

Suppose that the first case occurs. Following [MY1, Lemma 4], let

$$\Theta = \text{int}\{x \in \Phi : \varphi(x) = \varphi(y), \text{ some } x \neq y \in \Phi\}. \qquad (4.4)$$

We claim that $\partial\Theta \cap \partial\Phi \neq \emptyset$. Otherwise let $x \in \partial\Theta$. Then there exists $x \neq y \in \text{int}\Phi$ such that

$$\varphi(x) = \varphi(y)$$

and the tangents to φ at x and y are the same. By Lemma 4 there exist neighbourhoods U and V of x and y such that

$$\varphi(U) = \varphi(V),$$

contradicting the assumption that Θ is maximal. Hence there exists a point

$$z \in \partial\Theta \cap \partial\Phi$$

and, since φ is an embedding on $\partial\Phi$, $\varphi(z)$ is the image of some point of $\text{int}\Phi$, which is impossible.

Since the first case cannot occur, we deduce that there is a point where φ intersects itself transversely. Therefore some point of the image of φ lies outside P. This cannot occur because the image of φ coincides with the image of χ, which lies in P, so that the assumption that φ is not embedded leads to a contradiction.

We have thus found an embedded minimal surface $\varphi = \varphi_n$ in each of the manifolds $M_{\frac{1}{n}}$. By Theorem 3, a subsequence of (φ_n) converges to a minimal surface

$$f\colon F \to M \qquad (4.5.1)$$

such that

(4.5.2) F is a normalized slit domain homeomorphic to G,

(4.5.3) $f(\partial F) = \partial G$ and f is a monotone continuous map of degree 1 on each component of ∂F,

(4.5.4) if $\pi_2(M) = 0$, f(F) is isotopic to G, relative to ∂G.

We have assumed in the statement of the theorem that for each two-dimensional stratum H of ∂M the surface $\tilde{H}$ given by (C3) has non-negative mean curvature on a neighbourhood of H. Therefore if a point of intF is sent to ∂M, it follows by Lemma 1 that $\mathrm{im}(f) \subset \partial M$.

Then if f is not an embedding either im(f) is a closed surface or some non-empty proper subset of the curves in Γ is null-homologous in ∂M. Neither of these can happen because f sends $\pi_1(F)$ to $\pi_1(G)$ and G is incompressible in M.

We now assume that $f(\text{int}F) \subset \text{int}M$ and show that in this case also f is an embedding. Otherwise, since a point of intF cannot be sent to a point of Γ, there exist disjoint points $x, y \in \text{int}F$ satisfying

$$f(x) = f(y).$$

Then by Lemma 4 there exist neighbourhoods U and V of x and y such that either

$$f(U) = f(V)$$

or f(U) and f(V) have at least one transverse point of intersection. In the first case we define Θ as in (4.4), replacing Ψ and Φ by f and F, and argue that a point of intF is sent to a point of Γ, contradicting our assumption that $f(\text{int}F) \subset \text{int}M$. If f has a transverse point of self-intersection, so has Ψ for large enough n, contradicting the fact that Ψ is an embedding. $\square$

CHAPTER 5. EXAMPLES OF MINIMAL SURFACES WITH CERTAIN TOPOLOGICAL PROPERTIES.

In this chapter we construct examples which answer two questions about minimal surfaces in $\mathbb{R}^3$. Our constructions make use of the "bridge principle" for minimal surfaces, discussed in the Introduction: if F and G are stable minimal surfaces and γ is an arc joining ∂F to ∂G, then the bridge principle asserts that there exists a new minimal surface which consists of surfaces close to F and G together with a thin "bridge" running along γ. To justify this use of the bridge principle we apply the results of the previous chapters.

<u>Question</u> 1 [Meeks1, conjecture 5][Meeks2, Problem 1]. Given a set Γ of disjoint smooth Jordan curves on the standard 2-sphere S^2, such that Γ bounds two homeomorphic embedded compact connected minimal surfaces F and G in B^3, is there an isotopy of B^3 fixing Γ and taking F to G?

<u>Example</u> for Question 1. We shall first describe the construction of a counterexample informally in two steps, then prove that the surfaces are not isotopic in B^3 and then give a proof that such surfaces exist.

<u>Step</u> 1. In any system of polar coordinates on S^2, consider a pair of latitudes l_1 and l_2. l_1 and l_2 each bound a flat disc which is an embedded stable minimal surface (Figure 4(a)). Take a great circle g orthogonal to l_1 and l_2; this contains two arcs p and q which

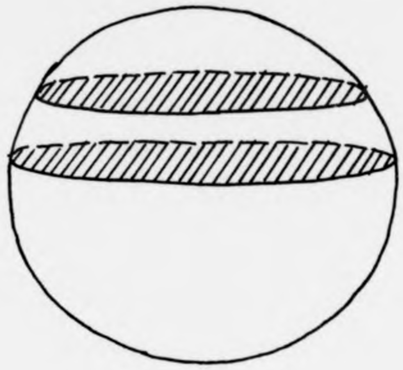

Figure 4(a).

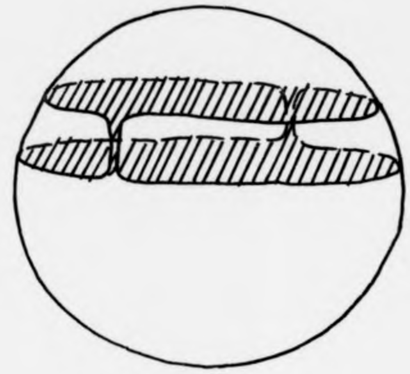

Figure 4(b).

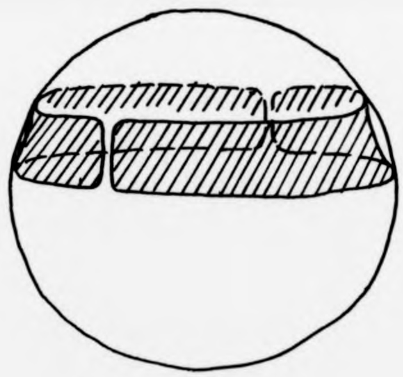

Figure 4(c).

join ℓ_1 to ℓ_2. Form a new pair C of disjoint smooth Jordan curves by erasing a short arc from ℓ_1 or ℓ_2 at each intersection with g and adding four arcs nearly parallel to p and q. If the bridge principle holds for C, C bounds a stable minimal annulus consisting of surfaces near the discs spanning l_1 and l_2 connected by bridges along p and q (Figure 4(b)). By results of Meeks and Yau, for example our Theorem 4, each component of C bounds an embedded stable minimal disc; each of these discs lies within the convex hull of its boundary and so they are disjoint (Figure 4(c)).

Step 2. Apply the construction of Step 1 to three pairs of latitudes, so as to obtain six Jordan curves on S^2 each of which bounds a disc in S^2 disjoint from the others. Denote these curves by a_i, b_i ($i = 1, 2, 3$) so that each pair (a_i, b_i) bounds a minimal annulus E_i from Step 1. Topologically, but not metrically, the annuli are arranged as in Figure 5.

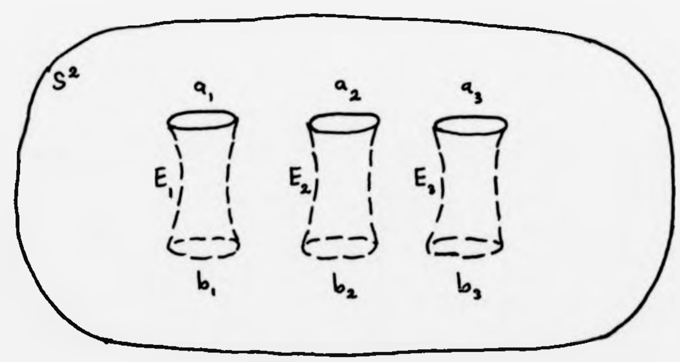

Figure 5.

To obtain an example where the surfaces F and G are surfaces of genus one with two boundary components, we connect these annuli by four further bridges α, β, γ and δ as shown in Figure 6.

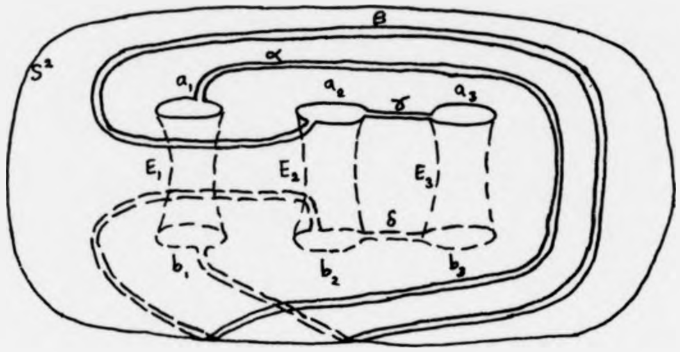

Figure 6.

The surface F consists of annuli close to E_1 and E_3 and discs spanning a_2 and b_2, connected by bridges. The surface G consists of annuli close to E_2 and E_3 and discs spanning a_1 and b_1, connected by bridges.

Each of F and G splits B^3 into a solid torus and a ball with two handles. For F the core of the solid torus is a trefoil knot (the topological picture is shown in Figure 7(a)) whereas for G it is unknotted (Figure 7(b)). Therefore there is no isotopy of B^3 taking F to G.

If we perform this construction leaving out the bridge δ we obtain surfaces F' and G', corresponding to F and G, each of

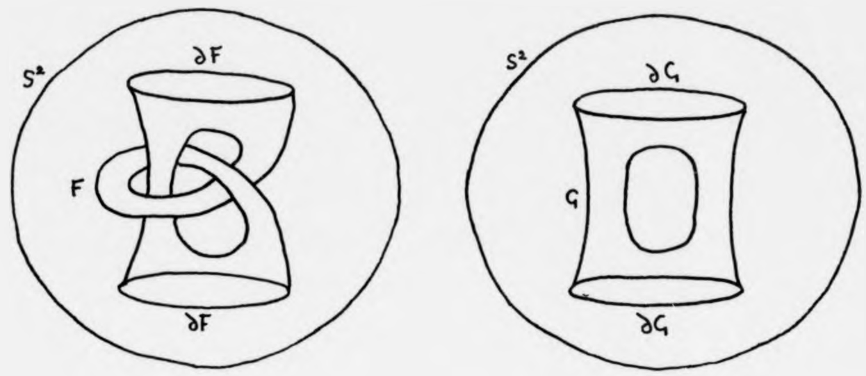

Figure 7(a). Figure 7(b).

which is a planar domain with three boundary components. Each of the three components of $\Gamma' = \partial F' = \partial G'$ bounds a disc in S^2 which is disjoint from the others; call this the disc <u>inside</u> that component. Regard B^3 as embedded in $\mathbb{R}^3$, and attach an unknotted 1-handle h to B^3 along the discs inside the components of Γ' which meet b_2 and b_3. Each of F' and G' splits B^3 into a ball and a ball with two handles. For F' the union of this ball with h is knotted whereas for G' it is unknotted. Therefore there is no isotopy of B^3 fixing Γ' and taking F' to G'.

We now prove the existence of the surfaces F and G by a method due to Meeks and Yau [MY2, Theorem 7]; the proof for F' and G' is similar. To prove that F exists we construct a three-dimensional submanifold M of $\mathbb{R}^3$ with piecewise-smooth boundary satisfying condition (C); M satisfies the additional condition that for each two-dimensional stratum H of ∂M the surface $\bar{H}$ to which it extends by (C3) has non-negative mean curvature on a neighbourhood of H. The set Γ of smooth Jordan curves may be taken to lie in $\partial M \cap S^2$ and M is a regular neighbourhood of a surface with the topological properties required of F. We then apply Theorem 4, with G of the theorem being the closure of either component of $\partial M \setminus \Gamma$.

The pairs of curves (a_1, b_1) and (a_3, b_3) are to be spanned by annuli. For each pair we start from the pair of latitudes ℓ_1 and ℓ_2 from which it was constructed in Step 1. For i = 1, 2 we take another two latitudes λ_i and μ_i, a short distance away from ℓ_i on either side. λ_i and μ_i bound an annular region A_i on S^2. A_i is to be part of the boundary of a ball B_i; ∂B_i will be piecewise-smooth and satisfy condition (C). The remainder of ∂B_i consists of spherical caps of radius much greater than 1 passing through λ_i and μ_i; we make B_1 disjoint from B_2.

The bridges connecting the two latitudes of Step 1 run along arcs p and q of a great circle g. For $\varepsilon < \frac{1}{2}$, the ε-neighbourhood of g has positive mean curvature with respect to the inward normal. We define the three-manifold L to be the union of B_1 and B_2 with the portions of a $\frac{1}{4}$-neighbourhood of g which lie inside S^2 and along p and q. (L is shaded in Figure 8.)

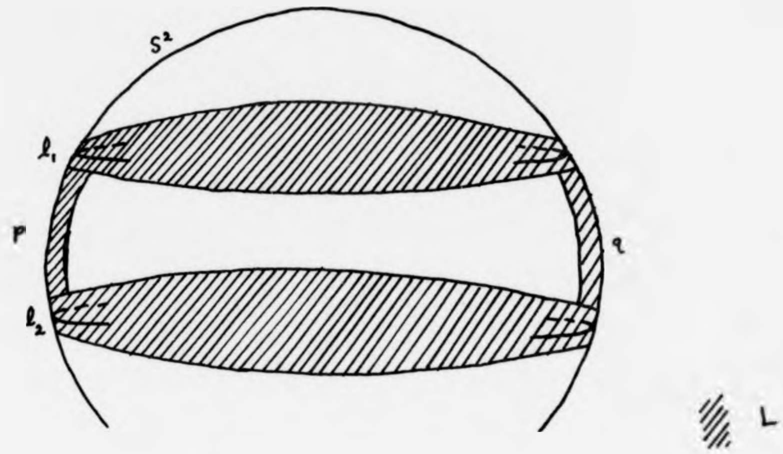

Figure 8.

We shall modify L to make its boundary satisfy condition (C). Consider the point x in which p or q intersects ∂B_i (i = 1 or 2). In a neighbourhood of x, L consists of part of B_i and part of the $\frac{1}{4}$-neighbourhood of g, and ∂L consists of a surface Σ which is part of a sphere of radius r_i, a surface Φ which is part of the $\frac{1}{4}$-neighbourhood of g and part of S^2 (Fig. 9). Consider a catenoid R whose axis of symmetry is the normal to Σ at x and which is also symmetrical about T, the plane tangent to Σ at x; there is a one-parameter family of such catenoids which differ by a homothety centred at x. Figure 9

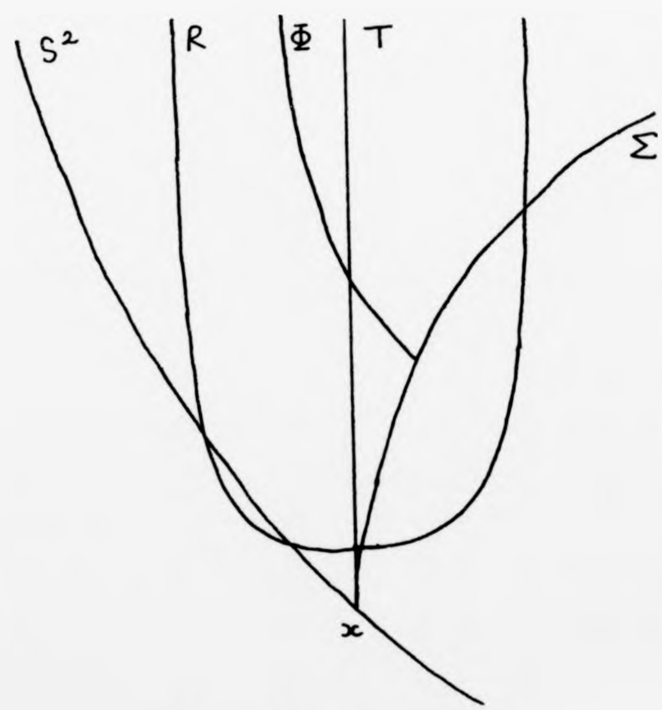

Fig. 9 (a).

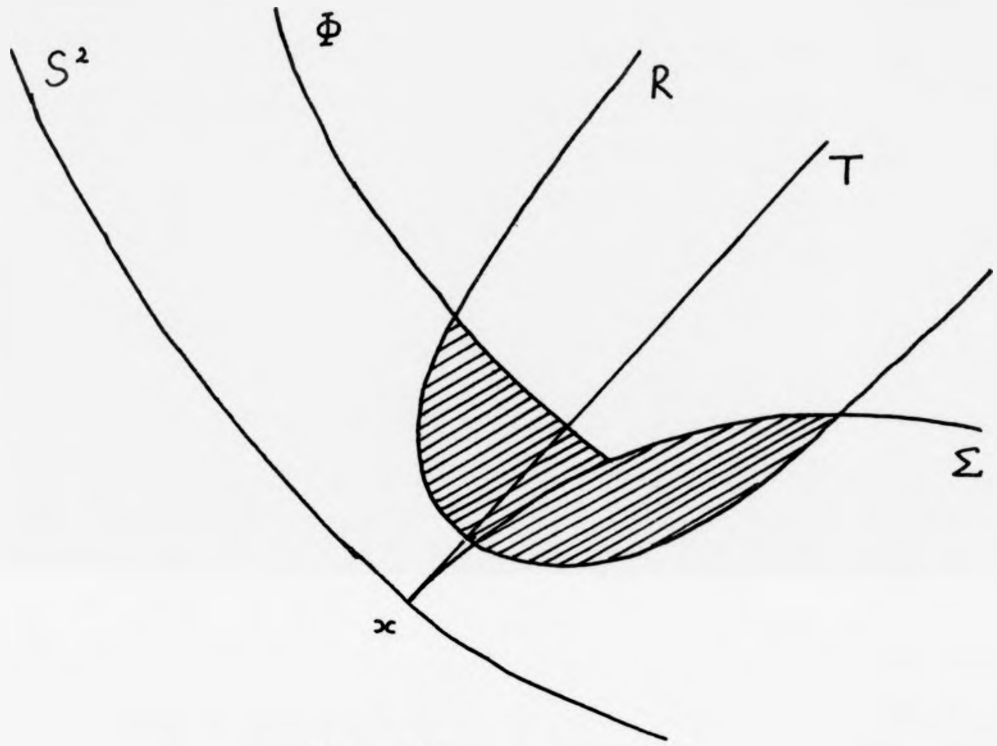

Fig. 9 (b).

shows the intersection of R with the plane P through the geodesic g. If the angle θ between T and the plane tangent to S^2 at x is small then, as we scale down R, $R \cap P$ begins to intersect g near x, and continues to do so for arbitrarily small values of the scale factor (Fig. 9 a). If θ is close to $\frac{\pi}{2}$, there is a neighbourhood of x in which $R \cap P$ does not intersect g for any scale factor (Fig. 9 b). It is the latter condition that we require, and we choose l_i, λ_i, μ_i and r_i so that it is satisfied. When R has been scaled down sufficiently, the component of $R \cap L$ which lies closest to x is a disc D. D, Σ and Φ cut off three balls from L, and we modify L by removing these three balls. (The portions to be removed are shaded in Fig. 9b.) With this modification at each such point ∂L satisfies condition (C).

We make one further modification to ∂L. Take a stable catenoid spanning two circles close to g, chosen so as to cut off small "channels" from L along the arcs p and q. By removing these "channels" from L we allow space for the bridges of Step 2.

The curves a_2 and b_2 are to be spanned by discs. Here we take new latitudes λ and μ a little outside ℓ_1 and ℓ_2 and form a single ball satisfying condition (C) in the same way as before. From this ball we remove the portion that lies between two planes parallel and close to the plane of g.

It remains to complete the construction of M by adding neighbourhoods of four curves on S^2 corresponding to the bridges α, β, γ and δ of Step 2. Where these meet the rest of M we intersect with a catenoid as before.

In the same way we construct a three-dimensional submanifold N of $\mathbb{R}^3$ with boundary satisfying condition (C) such that N is a regular neighbourhood of a surface with the topological properties required of G. It is possible to construct M and N so that $\partial M \cap S^2$ and $\partial N \cap S^2$ overlap sufficiently for the same set Γ of smooth Jordan curves to be chosen as ∂F and ∂G. The embedded minimal surfaces F and G then exist by an application of Theorem 4.

Remarks on Question 1. The author's first version of this proof made use of the geometric Dehn lemma and thus was restricted to surfaces of genus zero; he is grateful to S.T. Yau for drawing his attention to the result which is Theorem 4 of this thesis.

Meeks has shown that a compact connected minimal surface properly embedded in B^3 splits B^3 into two handlebodies [Meeks2, Proposition 2]. It follows that the answer to Question 1 is "Yes" if Γ consists of a single curve or if F and G are annuli [Meeks2, Corollary to Theorem 4]. We have described examples which show that the answer is "No" for the next simplest cases; evidently our examples can be modified to have higher genus or more boundary components. The question whether any properly embedded compact connected surface with boundary which splits B^3 into two handlebodies can be realised as a minimal surface appears to be harder.

Question 2 [Meeks1, conjecture 2][Nitschel, §910(b)]. Can a Jordan curve on the boundary of a convex set in $\mathbb{R}^3$ bound a minimal disc that is not embedded?

Example for Question 2. We take two disjoint Jordan curves on S^2 that bound stable minimal discs with interiors that intersect, and connect them by a bridge with boundary in S^2.

More precisely, let ℓ_i (i = 1, 2, 3) be latitudes close to the equator, ℓ_2 lying between ℓ_1 and ℓ_3. Form a Jordan curve $\Gamma_1 \subset S^2$ by connecting ℓ_1 and ℓ_2 with two arcs nearly parallel to a longitude. By the bridge principle, Γ_1 bounds a minimal disc D_1 consisting of discs close to the flat discs spanned by ℓ_1 and ℓ_2 connected by a thin bridge. D_1 does not minimize area, so Γ_1 bounds another minimal disc D_2 which lies close to S^2. Form a Jordan curve Γ_2 by connecting Γ_1 and ℓ_3 with two arcs nearly parallel to a longitude. Let D_3 be the flat disc bounded by ℓ_3. Then Γ_2 bounds a stable minimal disc D_4 consisting of discs close to D_1 and D_3 joined by a bridge, and Γ_2 also bounds a stable minimal disc D_5 consisting of discs close to D_2 and D_3 joined by a bridge. Let Γ_2' be a Jordan curve close to but not intersecting Γ_2. Then Γ_2 and Γ_2' bound stable minimal discs that intersect, and these may be connected by a bridge. As with the example for Question 1, we may prove that such a disc Δ exists by constructing a manifold satisfying condition (C), which in this case is immersed in $\mathbb{R}^3$ rather than embedded. (The reason for the disc D_3 is to allow us to use this method to attach the final bridge.)

Remarks on Question 2. We are not using the full strength of Theorem 4, since we do not need Δ to be embedded.

Having constructed an immersed minimal disc Δ with $\partial\Delta \subset S^2$ it is natural to ask whether such a disc can have a branch point. It does not seem possible to construct a disc with a branch point using the bridge principle.

The Jordan curve $\partial\Delta$ appears to bound at least five stable minimal discs, four obtained by the bridge principle together with the disc of least area, which appears to be a "ribbon" close to the curve C; it appears that only one of these fails to be embedded. There should be at least four unstable minimal discs as well. Given that a Jordan curve Γ bounds a minimal disc that is not embedded, how many other minimal discs must Γ bound? Meeks has shown that Γ must bound at least two embedded stable minimal discs [MY2, Theorem 4, Corollary 1].

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